



Design Document

Design: Hopper V2 Tank Plate

Document ID: M_HopperV2_Design_Tank_Plate

Version: 2.2

Date: October 25, 2026

Prepared by: Rufus Henschek

Contents

1	LOX In- and Outlet Position	2
1.1	Purpose of the Opening	2
1.2	Result	2
1.3	Analytical Model	3
1.3.1	Assumptions and Input Data	3
1.3.2	Plate under Uniform Pressure	3
1.3.3	Edge Restraint by the Tank Wall	4
1.3.4	Stress at the Edge of the Opening	4
1.3.5	Evaluation Quantities and Limits	4
1.3.6	Calculation Chain	5
1.4	Worked Example: Opening at $r_0 = 190$ mm	5
1.5	Finite Element Cross-Check	6
1.5.1	Formulation	8
1.5.2	Verification	9
1.5.3	Comparison and Limits	9
1.6	Derivations and Sources	9
1.6.1	Input Data	9
1.6.2	Plate under Uniform Pressure	10
1.6.3	Edge Restraint by the Tank Wall	10
1.6.4	Stress at the Edge of the Opening	11
1.6.5	Evaluation Quantities and Limits	11
2	Stiffening of the Bulkhead	13
2.1	Purpose and Design Basis	13
2.2	Result	13
2.3	Analytical Model	15
2.3.1	Layout and Design Variables	16
2.3.2	Load Path	16
2.3.3	Stringer Cross-Section	17
2.3.4	Radial Stringers	18
2.3.5	Ring Stringers	18
2.3.6	Plate between the Stringers	19
2.3.7	Coupled System	19
2.3.8	Mass	19
2.3.9	Radial Stringers Alone	19
2.3.10	Rings and Plate	20
2.3.11	Web Buckling	21
2.3.12	Centre Hub and Joints	22
2.3.13	Hub Wall between the Webs	23
2.3.14	Stringer Ends	23
2.3.15	Allowed Pressure at Room Temperature	23
2.3.16	Design Criteria and Failure Modes	24
2.4	Worked Calculation	24
2.4.1	Example: Radial Stringers Alone	24
2.4.2	Example: One Ring	24
2.4.3	Example: Web Buckling	25
2.4.4	Hub and Joints with a 2 mm Hub	26
2.4.5	Final Hub Wall ($t_h = 5$ mm)	26
2.4.6	Pressure at Room Temperature	27
2.5	Simulations	27
2.5.1	Radial Stringers over the Web Height	28

2.5.2	Ring Layouts	28
2.5.3	Web Buckling and Layout	31
2.5.4	Hub Variants and Joints	33
2.5.5	FE Check with Beam Elements	35
2.5.6	Final Layout	38
2.5.7	Ansys Model of the Complete Tank	39
2.6	Derivations and Sources	39
2.6.1	Mechanical Description	39
2.6.2	Radial Stringers Alone	42
2.6.3	Rings and Plate	42
2.6.4	Web Buckling	44
2.6.5	Centre Hub and Joints	44
2.6.6	Beam-Plate FE Model and Hub Wall	45
2.6.7	Allowed Pressure at Room Temperature	45
3	Manufacturing	47

List of Figures

1	FE mesh for the opening at $r_0 = 190$ mm	7
2	Hole-edge stress and deflection over the position of the opening	7
3	Layout sketch of the stringers	16
4	Cross-section of a stringer	17
5	Radial stringers alone: required number and stress	28
6	Example with no, one and two rings	29
7	Smallest web height and mass with up to three rings	29
8	All ring layouts: web height and mass	30
9	Lightest ring layouts	30
10	Buckling coefficient and allowable stress of the web	31
11	Allowable moment and lightest layouts with web buckling	32
12	Lightest layouts with web buckling	32
13	Hub variants and extra rings	33
14	Weld stresses, layout with 2 mm hub	34
15	FE mesh of the stiffened plate	36
16	FE deflection and plate stress	37
17	FE check: stringer stresses and mesh study	37
18	Final layout: stresses and deflection	38
19	Ansys, layout with 2 mm hub	40
20	Ansys, final design	40
21	CAD model of the final bulkhead	48

Abstract

This document describes the design of the common bulkhead in the propellant tank of Hopper V2. The bulkhead is a plate of stainless steel (316L) that divides the tank into a LOX and an LCH₄ compartment. The LOX line runs through the LCH₄ compartment and enters the LOX side through an opening in the plate.

In normal operation both compartments are pressurised to the same level, so the bulkhead carries almost no load. If one compartment loses its pressure, the full pressure difference acts on the plate, from either side. This emergency case is the design load. The bulkhead must then keep the two propellants apart: it may deform plastically and the tank may be lost, but there must be no breach between LOX and LCH₄. At the same time every gram of the bulkhead counts in the mass of the vehicle, so it is designed as light as this requirement allows.

The task is to find a position for the opening and a stiffening of the plate that keep deflection and stresses within the limits at the lowest possible mass. The design is made for the operating temperature and uses the higher strength of 316L at cryogenic temperature, without an additional safety factor. As a consequence, the allowed pressure difference at room temperature is lower than in operation.

The bulkhead is an experimental part with a target of TRL 4 to 5. The stress-strain curve of the sheet has not been measured and the laser welds cannot be inspected, so the analysis is backed by tests. They take place on a closed-off test site with the required test infrastructure: first with water at room temperature and reduced pressure, then with LN₂ at operating pressure, and finally with the real propellants.

Section 1 determines the position of the opening, Section 2 the stiffening of the plate, and Section 3 describes the manufacturing and the tests.

1 LOX In- and Outlet Position

1.1 Purpose of the Opening

The common bulkhead is an 8 mm plate made of cold-rolled 316L that divides the tank at mid-height of the cylindrical section. The upper compartment holds LOX, the lower one LCH₄. The LOX feed and drain line (outer diameter 25 mm, wall 1.5 mm) leaves the lower dome next to the LCH₄ port, runs through the LCH₄ compartment and is welded into an opening in the bulkhead. LOX is filled and drained through this opening.

Both compartments operate at 7 bar, so the bulkhead nominally sees no pressure difference. In the worst case one compartment loses its pressure, and the full difference $\Delta p = 0.7$ MPa acts on the plate. This can happen on either side, so the load can act in both directions. The 2 mm tank wall carries hardly any bending moment; the plate has to carry the bending itself and passes only shear forces into the wall.

An opening in a plate under bending is a notch: the stress at its edge is higher than in the undisturbed plate. How much higher depends on the bending state at the position of the opening. In addition, the LOX line is attached at this point, so the local deflection of the plate there is of interest as well. The aim of this section is therefore to find a radial position for the opening at which both the stress at the hole edge and the local deflection are low.

The study uses the plate alone, without stiffeners, as a reference. The opening is modelled as a free hole of 25 mm diameter without the tube. The tube would support the hole edge, so leaving it out is on the safe side. The hole is moved from the centre to the edge of the plate, and the result is compared for three edge conditions: simply supported, clamped, and elastically restrained by the tank wall.

1.2 Result

The opening is placed with its centre at

$$r_0 = 190 \text{ mm}, \quad d = 25 \text{ mm}, \quad (1)$$

measured from the centre of the plate. Table 1 compares this position with a central opening. The stresses are given relative to σ_0 , the bending stress at the centre of the solid, simply supported plate (Eq. (14)).

Table 1: Opening at $r_0 = 190$ mm compared with a central opening. Elastic: edge restraint by the tank wall, $\eta = 0.42$.

	simply supp.	elastic	clamped
σ_L/σ_0 , opening at centre	2.000	1.492	0.788
σ_L/σ_0 at $r_0 = 190$ mm, analytical	1.399	0.891	0.187
σ_L/σ_0 at $r_0 = 190$ mm, FEM	1.473	0.966	0.273
change of hole-edge stress vs. centre (analytical)	-30 %	-40 %	-76 %
$w(r_0)$ / max. deflection of the same case	45 %	43 %	27 %

- A central opening is the worst position for the simply supported and the elastically restrained plate. The bending moment is largest there and equal in both directions, so the full notch factor of 2 acts on the maximum moment.
- Moving the opening outwards lowers both the hole-edge stress and the local deflection. For the clamped plate the hole-edge stress has a minimum at $r \approx 193$ mm and rises again towards the clamped edge (Eq. (16)).

- The outer limit is not set by the curves but by the validity of the model: the distance from the hole centre to the plate edge must be at least $2.5 d$, i.e. $r_0 \leq 210.5 \text{ mm}$ (Eq. (15)). At $r_0 = 190 \text{ mm}$ this distance is $a - r_0 = 83 \text{ mm} = 3.3 d$.
- The FEM cross-check (Sec. 1.5) confirms the trend. Away from the centre the analytical model is 5 % to 8 % lower for the simply supported and elastic case, because it uses the moment at the hole centre only.

All absolute values refer to the unstiffened plate in linear theory and are not physical (Sec. 1.3.5). Only the ratios are used for the decision.

1.3 Analytical Model

This section lists only the formulas that are used in the calculation. Their derivation and the literature sources are given in Sec. 1.6.

1.3.1 Assumptions and Input Data

- The bulkhead is a thin circular Kirchhoff plate under uniform pressure, linear theory, without stiffeners.
- The moments of the solid plate at the position of the hole are taken as the nominal field; the notch effect of the small hole is added on top.
- The opening is a free hole without the tube.

Table 2: Input data.

Quantity	Symbol	Value
Young's modulus (1.4404, 20 °C)	E	200 000 MPa
Poisson's ratio	ν	0.3
Pressure difference	p	0.7 MPa
Plate radius	a	273 mm
Plate thickness	s_P	8 mm
Mean radius of the tank wall	R_m	274 mm
Wall thickness	t_T	2 mm
Diameter of the opening	d	25 mm

1.3.2 Plate under Uniform Pressure

With the plate stiffness

$$K = \frac{E s_P^3}{12(1 - \nu^2)} \quad (2)$$

and $\rho = r/a$, the bending moments are

$$\text{simply supported: } M_r = \frac{p a^2}{16} (3 + \nu) (1 - \rho^2), \quad M_\varphi = \frac{p a^2}{16} [(3 + \nu) - (1 + 3\nu) \rho^2], \quad (3)$$

$$\text{clamped: } M_r = \frac{p a^2}{16} [(1 + \nu) - (3 + \nu) \rho^2], \quad M_\varphi = \frac{p a^2}{16} [(1 + \nu) - (1 + 3\nu) \rho^2], \quad (4)$$

and the deflections are

$$w_{ss}(\rho) = \frac{p a^4}{64 K} \left[\frac{5 + \nu}{1 + \nu} - 2 \frac{3 + \nu}{1 + \nu} \rho^2 + \rho^4 \right], \quad (5)$$

$$w_{cl}(\rho) = \frac{p a^4}{64 K} (1 - \rho^2)^2. \quad (6)$$

For the elastically restrained edge with the degree of restraint η (Eq. (11)):

$$M_{el} = M_{ss} - \eta \frac{p a^2}{8} \quad (\text{for } M_r \text{ and } M_\varphi), \quad w_{el} = w_{ss} + \eta (w_{cl} - w_{ss}). \quad (7)$$

1.3.3 Edge Restraint by the Tank Wall

$$\beta = \left[\frac{3(1-\nu^2)}{R_m^2 t_T^2} \right]^{1/4}, \quad B_M = \frac{E t_T^3}{12(1-\nu^2)}, \quad (8)$$

$$k_M = 4\beta B_M, \quad (9)$$

$$k_P = \frac{K(1+\nu)}{a}, \quad (10)$$

$$\eta = \frac{k_M}{k_M + k_P}, \quad 0 \leq \eta \leq 1. \quad (11)$$

k_M is the rotational stiffness of the tank wall (both halves) and k_P that of the plate, each per unit length of the circumference. $\eta = 0$ means a simply supported edge, $\eta = 1$ a clamped edge.

1.3.4 Stress at the Edge of the Opening

$$\sigma_r = \frac{6 M_r}{s_P^2}, \quad \sigma_\varphi = \frac{6 M_\varphi}{s_P^2}, \quad (12)$$

$$\sigma_L = |\sigma_r + \sigma_\varphi| + k |\sigma_r - \sigma_\varphi|, \quad k = \frac{2(1+\nu)}{3+\nu} = 0.788. \quad (13)$$

1.3.5 Evaluation Quantities and Limits

Reference values (solid, simply supported plate at the centre):

$$\sigma_0 = \frac{6}{s_P^2} \frac{(3+\nu) p a^2}{16}, \quad w_0 = \frac{5+\nu}{1+\nu} \frac{p a^4}{64 K}. \quad (14)$$

Validity limit of the hole-edge formula, distance from the hole centre to the plate edge at least $2.5d = 5r_L$ (hole radius $r_L = d/2$):

$$r_0 \leq a - 2.5d = 210.5 \text{ mm}. \quad (15)$$

This limit refers to the plate edge only. The clearance between the opening and the stringers in Sec. 2 is a separate criterion with a different distance, $2.5r_L$ (Eq. (24), Sec. 1.6.5).

Position where the first term of Eq. (13) vanishes (minimum of the hole-edge stress):

$$\rho^2 = \frac{(3+\nu) - 2\eta}{2(1+\nu)} \Rightarrow \begin{cases} \eta = 1 : & r = 193 \text{ mm} \\ \eta = 0.42 : & r = 266 \text{ mm} \\ \eta = 0 : & \rho^2 > 1, \text{ no minimum inside the plate.} \end{cases} \quad (16)$$

In-plane load on the plate from the radial expansion of the pressurised wall half (upper bound):

$$\delta = \frac{p(1-\nu/2) R_m^2}{E t_T}, \quad Q \leq 4\beta^3 B_M \delta, \quad \sigma_m = \frac{Q}{s_P}. \quad (17)$$

For the unstiffened plate $w_0/s_P = 3.3$, which is far outside linear plate theory. The absolute values are therefore not physical, and only the ratios over r_0 are used.

1.3.6 Calculation Chain

For every position r_0 the following steps are carried out:

1. Plate stiffness K (Eq. (2)) and degree of restraint η (Eqs. (8)–(11)).
2. Moments M_r , M_φ at $\rho = r_0/a$ for the three edge conditions (Eqs. (3), (4), (7)).
3. Nominal stresses (Eq. (12)) and hole-edge stress σ_L (Eq. (13)).
4. Deflection $w(r_0)$ (Eqs. (5), (6), (7)).
5. Normalisation with σ_0 , w_0 (Eq. (14)) and check of the distance from the hole centre to the plate edge (Eq. (15)).

The chain is implemented in the script `S1_Bohrungslage.py`, which prints all intermediate values and plots σ_L/σ_0 and w/w_0 over r_0 .

1.4 Worked Example: Opening at $r_0 = 190$ mm

Geometry and plate.

$$\begin{aligned}\rho &= \frac{190}{273} = 0.6960, & \rho^2 &= 0.4844, & \rho^4 &= 0.2346 \\ K &= \frac{200000 \cdot 8^3}{12(1 - 0.3^2)} = 9.377 \times 10^6 \text{ N} \cdot \text{mm} \\ \frac{p a^2}{16} &= \frac{0.7 \cdot 273^2}{16} = 3260.64 \text{ N} \cdot \text{mm/mm}, & \frac{p a^2}{8} &= 6521.29 \text{ N} \cdot \text{mm/mm} \\ \frac{p a^4}{64 K} &= \frac{0.7 \cdot 273^4}{64 \cdot 9.377 \cdot 10^6} = 6.479 \text{ mm} \\ \sigma_0 &= \frac{6}{8^2} \cdot 3.3 \cdot 3260.64 = 1008.76 \text{ MPa}, & w_0 &= \frac{5.3}{1.3} \cdot 6.479 = 26.413 \text{ mm} \quad (w_0/s_P = 3.30)\end{aligned}$$

Edge restraint.

$$\begin{aligned}\beta &= \left[\frac{3(1 - 0.3^2)}{274^2 \cdot 2^2} \right]^{1/4} = 0.05491/\text{mm} \quad (1/\beta = 18.21 \text{ mm}) \\ B_M &= \frac{200000 \cdot 2^3}{12(1 - 0.3^2)} = 1.4652 \times 10^5 \text{ N} \cdot \text{mm} \\ k_M &= 4 \cdot 0.05491 \cdot 1.4652 \cdot 10^5 = 3.218 \times 10^4 \text{ N} \cdot \text{mm/mm} \\ k_P &= \frac{9.377 \cdot 10^6 \cdot 1.3}{273} = 4.465 \times 10^4 \text{ N} \cdot \text{mm/mm} \\ \eta &= \frac{3.218}{3.218 + 4.465} = 0.419, & M_a &= -0.419 \cdot 6521.29 = -2731.37 \text{ N} \cdot \text{mm/mm}\end{aligned}$$

The stiffnesses k_M and k_P are per unit length of the circumference and per radian.

Moments and stresses at r_0 . Simply supported (Eq. (3)):

$$\begin{aligned}M_r &= 3260.64 \cdot 3.3 \cdot (1 - 0.4844) = 5548.19 \text{ N} \cdot \text{mm/mm} \\ M_\varphi &= 3260.64 \cdot (3.3 - 1.9 \cdot 0.4844) = 7759.31 \text{ N} \cdot \text{mm/mm}\end{aligned}$$

Elastic (Eq. (7)):

$$\begin{aligned}M_r &= 5548.19 - 2731.37 = 2816.82 \text{ N} \cdot \text{mm/mm} \\ M_\varphi &= 7759.31 - 2731.37 = 5027.95 \text{ N} \cdot \text{mm/mm}\end{aligned}$$

Clamped (Eq. (4)):

$$M_r = 3260.64 \cdot (1.3 - 3.3 \cdot 0.4844) = -973.10 \text{ N} \cdot \text{mm/mm}$$

$$M_\varphi = 3260.64 \cdot (1.3 - 1.9 \cdot 0.4844) = 1238.02 \text{ N} \cdot \text{mm/mm}$$

With $\sigma = 6M/s_P^2 = 0.09375 M$ (Eq. (12)) and Eq. (13), simply supported:

$$\sigma_r = 520.14 \text{ MPa}, \quad \sigma_\varphi = 727.44 \text{ MPa}$$

$$\sigma_L = |520.14 + 727.44| + 0.788 \cdot |520.14 - 727.44| = 1247.58 + 163.32 = 1410.90 \text{ MPa}$$

$$\sigma_L/\sigma_0 = 1410.90/1008.76 = 1.399$$

The other two cases follow in the same way (Table 3). The difference term $k|\sigma_r - \sigma_\varphi| = 163.32 \text{ MPa}$ is the same in all three cases, because the edge moment adds equally to M_r and M_φ .

Deflection at r_0 .

$$w_{ss} = 6.479 \cdot \left(\frac{5.3}{1.3} - 2 \cdot \frac{3.3}{1.3} \cdot 0.4844 + 0.2346 \right) = 12.001 \text{ mm}$$

$$w_{cl} = 6.479 \cdot (1 - 0.4844)^2 = 1.723 \text{ mm}$$

$$w_{el} = 12.001 + 0.419 \cdot (1.723 - 12.001) = 7.696 \text{ mm}$$

Table 3: Worked example, opening at $r_0 = 190 \text{ mm}$ (analytical model).

	simply supported	elastic	clamped
M_r [N · mm/mm]	5548.19	2816.82	-973.10
M_φ [N · mm/mm]	7759.31	5027.95	1238.02
σ_r [MPa]	520.14	264.08	-91.23
σ_φ [MPa]	727.44	471.37	116.06
$ \sigma_r + \sigma_\varphi $ [MPa]	1247.58	735.45	24.84
$k \sigma_r - \sigma_\varphi $ [MPa]	163.32	163.32	163.32
σ_L [MPa]	1410.90	898.77	188.16
σ_L/σ_0	1.399	0.891	0.187
σ_L/σ_0 , opening at centre	2.000	1.492	0.788
$w(r_0)$ [mm]	12.001	7.696	1.723
$w(r_0)/w_0$	0.454	0.291	0.065
$w(r_0)$ / max. deflection of the same case	0.454	0.426	0.266

Checks.

- Distance from the hole centre to the plate edge: $a - r_0 = 83 \text{ mm} = 3.32 d \geq 2.5 d$ (Eq. (15)).
- In-plane force (Eq. (17)): $\delta = 0.7 \cdot 0.85 \cdot 274^2 / (200000 \cdot 2) = 0.112 \text{ mm}$, $Q \leq 4 \cdot 0.05491^3 \cdot 1.4652 \cdot 10^5 \cdot 0.112 = 10.84 \text{ N/mm}$, $Q/s_P = 1.35 \text{ MPa}$, i.e. about 0.1 % of σ_L .

The values are reproduced by `S1_Bohrungslage.py`; the FEM values at 190 mm in Table 1 come from `S1b_FEM_Bohrung.py`.

1.5 Finite Element Cross-Check

The analytical model relies on two simplifications: the moment of the solid plate is taken at the hole centre, and the notch factor of an infinite plate is put on top. A small plate FE model (script `S1b_FEM_Bohrung.py`) solves the plate with the hole directly and checks both. It uses the same input data (Table 2) and the same three edge conditions.

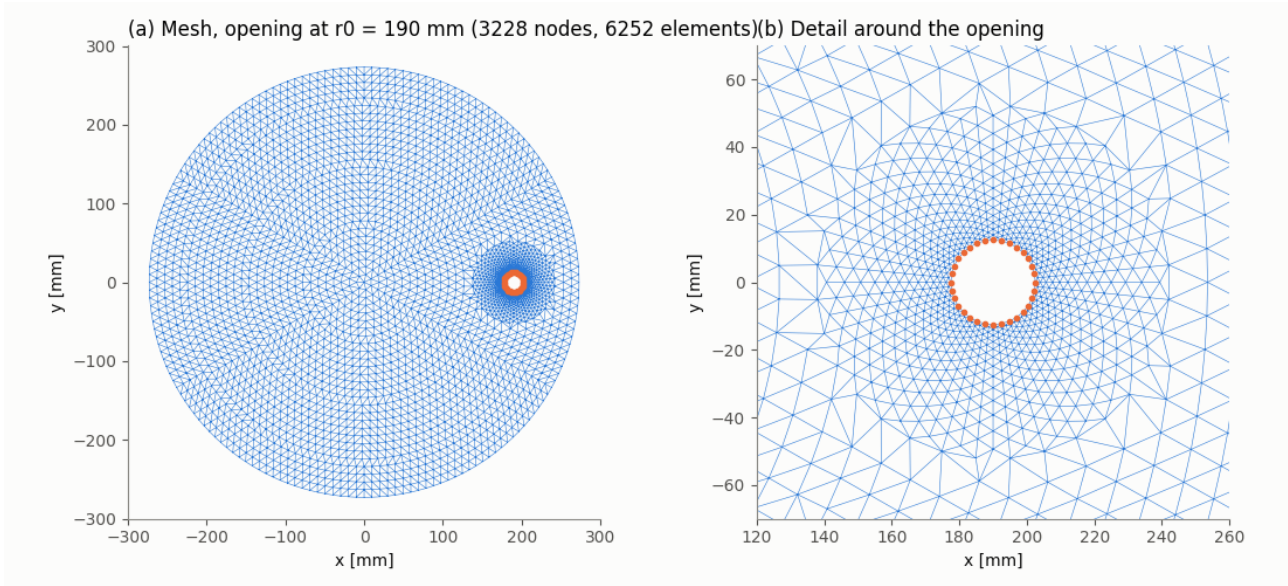


Figure 1: FE mesh for the opening at $r_0 = 190$ mm: (a) full plate, (b) detail around the opening. Concentric rings around the plate centre and the opening, Delaunay triangulation, 32 elements on the hole circumference.

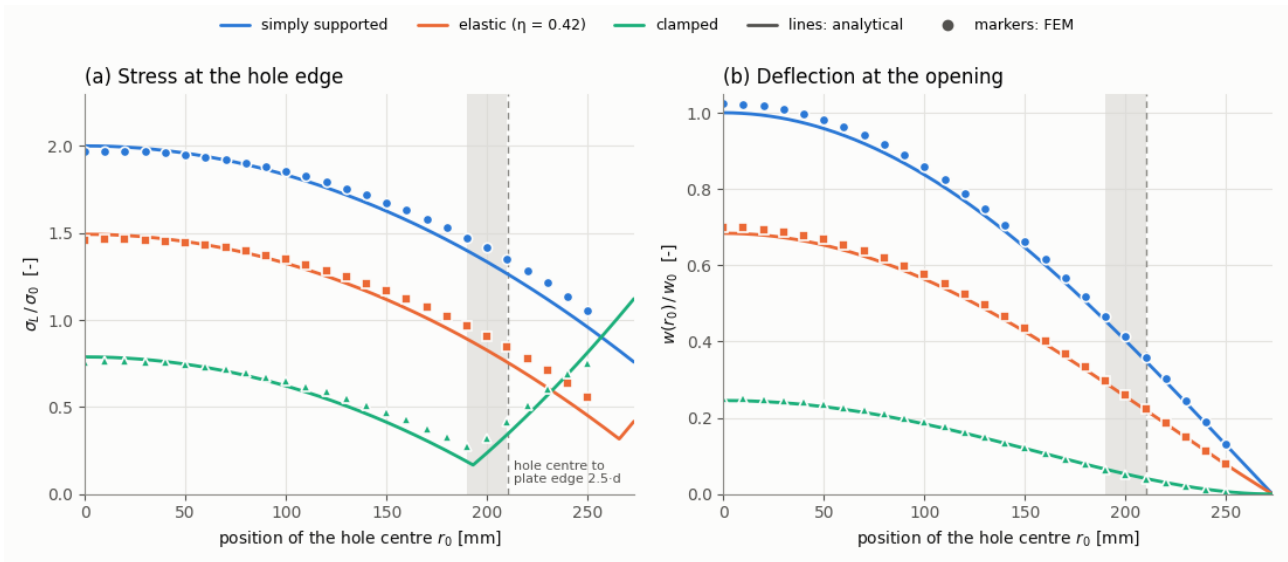


Figure 2: (a) Stress at the hole edge and (b) plate deflection at the opening versus the position r_0 of the hole centre. Lines: analytical model (Sec. 1.3); markers: FEM. Grey band: 190 mm to 210 mm; dashed line: limit $r_0 = a - 2.5d$ (distance from the hole centre to the plate edge, Eq. (15)).

1.5.1 Formulation

Element. The plate is discretised with the Discrete Kirchhoff Triangle (DKT) after Batoz, as presented in [L]. Each corner node carries the deflection and the two slopes,

$$\mathbf{u}_i = \left[w_i, \left(\frac{\partial w}{\partial x} \right)_i, \left(\frac{\partial w}{\partial y} \right)_i \right]^T. \quad (18)$$

[L] uses the rotations β_x, β_y instead of the slopes. Under the Kirchhoff hypothesis both describe the same quantity and differ only in sign and orientation.

Slope field. The slope field $\mathbf{s} = [\partial w / \partial x, \partial w / \partial y]^T$ is interpolated quadratically over six points (three corners, three mid-sides) with the quadratic shape functions N_a [L]:

$$\mathbf{s}(x, y) = \sum_{a=1}^6 N_a(x, y) \mathbf{s}_a. \quad (19)$$

The mid-side slopes are expressed by the corner values. Along each edge ij of length l_{ij} , w is cubic, which gives the slope along the edge at the mid-side k . The slope normal to the edge varies linearly [L]:

$$\left(\frac{\partial w}{\partial s} \right)_k = \frac{3}{2l_{ij}} (w_j - w_i) - \frac{1}{4} \left[\left(\frac{\partial w}{\partial s} \right)_i + \left(\frac{\partial w}{\partial s} \right)_j \right], \quad \left(\frac{\partial w}{\partial n} \right)_k = \frac{1}{2} \left[\left(\frac{\partial w}{\partial n} \right)_i + \left(\frac{\partial w}{\partial n} \right)_j \right]. \quad (20)$$

This gives a linear map $\mathbf{s}_a = \mathbf{S} \mathbf{u}_e$ from the nine element DOFs to the twelve slope values.

Curvature, moments, stiffness.

$$\boldsymbol{\kappa} = \left[\frac{\partial s_x}{\partial x}, \frac{\partial s_y}{\partial y}, \frac{\partial s_x}{\partial y} + \frac{\partial s_y}{\partial x} \right]^T = \mathbf{B} \mathbf{S} \mathbf{u}_e, \quad \mathbf{M} = -\mathbf{D}_b \boldsymbol{\kappa}, \quad \mathbf{D}_b = K \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}, \quad (21)$$

$$\mathbf{k}_e = \int_{A_e} (\mathbf{B}\mathbf{S})^T \mathbf{D}_b (\mathbf{B}\mathbf{S}) \, dA \approx \sum_{g=1}^3 \frac{A_e}{3} (\mathbf{B}\mathbf{S})_g^T \mathbf{D}_b (\mathbf{B}\mathbf{S})_g. \quad (22)$$

The curvature relation corresponds to [L] and the stiffness to [L]. [L] integrates analytically; here a three-point Gauss rule is used, which is exact because $\mathbf{B}$ is linear and the integrand quadratic. The pressure is lumped to the corners, $f_{w,i} = p A_e / 3$.

System and boundary conditions. The element matrices are assembled into $\mathbf{K} \mathbf{u} = \mathbf{F}$ (sparse solver). At the outer edge nodes:

- simply supported: $w = 0$;
- clamped: $w = \partial w / \partial x = \partial w / \partial y = 0$;
- elastic: $w = 0$, plus a rotational spring on the slope normal to the edge, $\mathbf{k}_i = k_M \Delta s \mathbf{n} \mathbf{n}^T$, with k_M from Eq. (9), the node spacing Δs and the outward normal $\mathbf{n}$.

Stress recovery. The moments are evaluated at the corners of the elements and averaged at each node on the hole edge. From the principal moments $M_{1,2}$ the stress is

$$\sigma = \frac{6 \max(|M_1|, |M_2|)}{s_P^2},$$

and σ_L is the maximum over the hole circumference.

Mesh. The mesh consists of concentric rings around the plate centre (element size 10 mm) and finer rings around the opening (32 elements on the circumference). The radial element size at the hole edge is 0.3 times the circumferential size, because the stress gradient there is radial. Both ring systems are joined by a Delaunay triangulation, and elements inside the hole are removed.

1.5.2 Verification

Table 4: Verification of the FE model.

	simply supp.	elastic	clamped
<i>Check 1: solid plate vs. Eqs. (5)–(7)</i>			
w at centre, FEM / analyt. [mm]	26.405 / 26.413	18.059 / 18.064	6.482 / 6.479
M at centre, FEM / analyt. [N · mm/mm]	10766 / 10760	8036 / 8029	4248 / 4239
<i>Check 2: hole at centre vs. exact annular plate</i>			
σ_L , FEM / exact [MPa]	1983.3 / 1996.0	1472.1 / 1481.2	767.1 / 771.2
deviation	−0.6 %	−0.6 %	−0.5 %
<i>Check 3: mesh study at $r_0 = 190$ mm, σ_L/σ_0</i>			
coarse (16 mm, 24 on hole)	1.470	0.965	0.274
used (10 mm, 32 on hole)	1.473	0.966	0.273
fine (7 mm, 48 on hole)	1.476	0.967	0.272

The exact reference for check 2 is the annular plate with a free inner edge at $r_L = d/2$, solved with all four constants of [TM4]. Without the radial refinement at the hole edge, the mesh study gives $\sigma_L/\sigma_0 = 1.409$ instead of 1.473, i.e. about 4 % too low.

1.5.3 Comparison and Limits

FEM and analytical model agree in trend and in the position of the minimum of the clamped case (Fig. 2). At the centre the deviation is below 2 %. Away from the centre the FEM gives 5 % to 8 % higher hole-edge stresses for the simply supported and elastic case. The analytical model uses the moment at the hole centre, while the moment actually varies over the 25 mm of the hole, and the inner flank (closer to the plate centre) sees a higher value. For the clamped case near its minimum the relative deviation is large but the absolute value remains small. The hole raises the deflection at the opening by about 2 % to 3 %.

Both models share the same assumptions: thin plate without transverse shear, linear theory, ideal edge conditions, unstiffened plate. The agreement therefore verifies the solution of the plate equations, not the plate model itself. With $d/s_P = 3.1$ the real notch factor may be somewhat higher than Goodier’s thin-plate value. None of these assumptions changes the ranking of the positions, which is what the decision is based on.

1.6 Derivations and Sources

This section shows where the formulas of Sec. 1.3 come from. The location in a source is given as printed page, with the PDF page of the library file in parentheses; a key without location refers back to the passage cited just before. Steps that are not taken from the literature but derived here are marked as such.

1.6.1 Input Data

$E = 200\,000$ MPa is the value for 1.4404 at 20 °C [EN10088]. For metallic materials Wiedemann takes $\nu = 0.3$ [W]; the coefficients of the hole-edge formula (Sec. 1.6.4) are given for this value.

1.6.2 Plate under Uniform Pressure

Plate theory and stiffness. The bulkhead is treated as a thin Kirchhoff plate [TM4]; with $a/s_P = 34$ it is thin. The plate stiffness Eq. (2) is defined in [TM4]; Wiedemann gives the same expression as the bending stiffness B of the isotropic plate [W].

Moments from the deflection. For axisymmetric deflection the moments follow from the elasticity law of the plate in polar coordinates [TM4]:

$$M_r = -K \left(\frac{d^2 w}{dr^2} + \frac{\nu}{r} \frac{dw}{dr} \right), \quad M_\varphi = -K \left(\nu \frac{d^2 w}{dr^2} + \frac{1}{r} \frac{dw}{dr} \right). \quad (23)$$

The deflections Eq. (5) (simply supported) and Eq. (6) (clamped) are the solutions in [TM4] and [TM4]. Their derivatives are

$$\begin{aligned} \text{simply supported: } w' &= \frac{pr}{16K} \left[r^2 - \frac{3+\nu}{1+\nu} a^2 \right], & w'' &= \frac{p}{16K} \left[3r^2 - \frac{3+\nu}{1+\nu} a^2 \right], \\ \text{clamped: } w' &= -\frac{pr}{16K} (a^2 - r^2), & w'' &= -\frac{p}{16K} (a^2 - 3r^2). \end{aligned}$$

Inserting them into Eq. (23) gives Eqs. (3) and (4).

Elastic edge restraint (superposition). The term $C_2 r^2$ of the general solution [TM4] describes the solid plate under a uniform edge moment M_a . With Eq. (23) it gives $M_r = M_\varphi = M_a$ everywhere and

$$w_{M_a} = \frac{M_a (a^2 - r^2)}{2K(1+\nu)}.$$

The clamped plate is the simply supported plate plus $M_a = -p a^2/8$. An elastic edge carries only the part $M_a = -\eta p a^2/8$ (Sec. 1.6.3). Since the problem is linear, moments and deflections superpose, which gives Eq. (7) (derived here).

1.6.3 Edge Restraint by the Tank Wall

Stiffnesses of the wall. The membrane stiffnesses of an isotropic sheet are [W]

$$c_{11} = c_{22} = \frac{E t_T}{1 - \nu^2}, \quad c_{12} = \frac{\nu E t_T}{1 - \nu^2},$$

and its bending stiffness is [W]

$$B_M = c_{44} = \frac{E t_T^3}{12(1 - \nu^2)}.$$

Wiedemann writes the shell equation for the general orthotropic wall. There the axial bending stiffness is $B_x = b_{44} = c_{44}$ [W], and the circumferential stiffness without transverse restraint is $D_y = 1/c_{22}^* = c_{22} - c_{12}^2/c_{11}$ [W]. For the isotropic wall:

$$B_x = B_M, \quad D_y = \frac{E t_T}{1 - \nu^2} (1 - \nu^2) = E t_T.$$

Wall as a beam on an elastic foundation. For rotationally symmetric loading the radial displacement $u(x)$ of the wall satisfies [W]

$$B_x u'''' + \frac{D_y}{R_m^2} u = p_w.$$

Wiedemann notes that this is the equation of a beam with bending stiffness B_x on an elastic foundation with the spring constant D_y/R_m^2 [W]. (Wiedemann uses w for the radial displacement; u is used here to keep w for the plate deflection.) The solution that decays away from the edge is [W]

$$u = u_p \left[1 + \left(C_1 \cos \frac{\mu x}{R_m} + C_2 \sin \frac{\mu x}{R_m} \right) e^{-\mu x/R_m} \right], \quad \mu = \left(\frac{R_m^4 D_y}{4 B_x} \right)^{1/4}.$$

With $\beta = \mu/R_m$ and $B_x = B_M$, $D_y = E t_T$:

$$\beta = \left(\frac{D_y}{4 B_x R_m^2} \right)^{1/4} = \left(\frac{E t_T \cdot 12 (1 - \nu^2)}{4 E t_T^3 R_m^2} \right)^{1/4} = \left[\frac{3 (1 - \nu^2)}{R_m^2 t_T^2} \right]^{1/4},$$

which is Eq. (8). The decay length is $1/\beta = 18.2$ mm, much shorter than each wall half (about 500 mm), so both halves act as semi-infinite.

Rotational stiffness of the wall (derived here). $[\mathbf{W}]$ gives the solution, but not the edge stiffness; it is derived here from the homogeneous part of Eq. (4.2-67). In its plane the plate is much stiffer than the wall, so it holds the wall edge radially: $u(0) = 0$, hence $C_1 = 0$ and

$$u(x) = C e^{-\beta x} \sin(\beta x), \quad u'(0) = \beta C, \quad u''(0) = -2 \beta^2 C.$$

With the curvature $\kappa_x = -u''$ $[\mathbf{W}]$, the edge moment is $m_x(0) = -B_M u''(0) = 2 \beta^2 B_M C$ and the edge rotation is $\varphi(0) = u'(0) = \beta C$. One wall half therefore has the rotational stiffness $m_x/\varphi = 2 \beta B_M$; both halves together give Eq. (9).

Rotational stiffness of the plate and degree of restraint (derived here). From w_{M_a} above, the edge moment M_a rotates the plate edge by $\varphi = -M_a a/[K(1 + \nu)]$, which gives Eq. (10). The simply supported plate under p rotates its edge by $\varphi_{ss} = w'_{ss}(a) = -p a^3/[8K(1 + \nu)]$ (derivative above). The wall answers the total edge rotation with $M_a = k_M \varphi$:

$$M_a = k_M \left[-\frac{p a^3}{8K(1 + \nu)} - \frac{M_a a}{K(1 + \nu)} \right] \Rightarrow M_a = -\frac{p a^2}{8} \frac{k_M}{k_M + k_P} = -\eta \frac{p a^2}{8},$$

which is Eq. (11).

1.6.4 Stress at the Edge of the Opening

Validity. The notch solutions apply to openings that are small compared with the plate, so that the stresses at the hole do not depend on the outer edges $[\mathbf{W}]$. Here $d/2a \approx 0.05$.

Goodier's solution. For an open circular hole in a plate under bending, Goodier's solution gives the stresses around the hole $[\mathbf{NACA740}]$. For a wide plate under cylindrical bending the maximum is $[\mathbf{NACA740}]$

$$\sigma_L = \frac{(1 + \nu)(5 - \nu)}{3 + \nu} \sigma_1 \quad (\sigma_2 = \nu \sigma_1),$$

i.e. $1.85 \sigma_1$ for steel. The test in $[\mathbf{NACA740}]$ measured 1.85.

General biaxial bending. For an elliptical hole Wiedemann gives the extreme edge stresses for $\nu = 0.3$ as $\sigma_x = (1 + 0.79 b/a) \sigma_{x\infty} + 0.21 \sigma_{y\infty}$ $[\mathbf{W}]$. For a circular hole ($a = b$) and the principal stresses σ_1, σ_2 :

$$\sigma_L = 1.79 \sigma_1 + 0.21 \sigma_2 = (\sigma_1 + \sigma_2) + 0.79 (\sigma_1 - \sigma_2).$$

Taking the larger value of both flank points and writing it with σ_r, σ_φ gives Eq. (13). The factor k follows exactly from the NACA result above: with $\sigma_2 = \nu \sigma_1$, $(1 + \nu) + k(1 - \nu) = (1 + \nu)(5 - \nu)/(3 + \nu)$, hence $k = 2(1 + \nu)/(3 + \nu) = 0.788$, which Wiedemann rounds to 0.79 (derived here).

Checks. For equal bending in both directions ($\sigma_r = \sigma_\varphi$, plate centre) Eq. (13) gives $\sigma_L = 2 \sigma$ $[\mathbf{W}]$. For cylindrical bending it gives $1.85 \sigma_1$ $[\mathbf{W}]$.

1.6.5 Evaluation Quantities and Limits

Reference values. Eq. (14) is Eq. (3) with Eq. (12) and Eq. (5), both at $\rho = 0$. The deflection at the opening is taken from the solid plate at $r = r_0$; the FEM shows that the hole raises it by only 2 % to 3 % (Sec. 1.5).

Distance from the hole centre to the plate edge. Goodier's solution holds for finite plates without appreciable error as long as the distance from the hole centre to the plate edge is at least $2.5d$ [NACA740]. Wiedemann states for the in-plane case that the disturbance of a hole has practically decayed at a distance of $5r_L = 2.5d$ [W]. This gives Eq. (15).

Two different distances to the opening are used in this document:

- hole centre to plate edge $\geq 2.5d = 5r_L = 62.5 \text{ mm}$ (Eq. (15)): validity limit of Goodier's solution, which is derived for an infinite plate;
- hole centre to stringer centre line $\geq 2.5r_L = 1.25d = 31.25 \text{ mm}$ (Eqs. (24), (25)): in bending, the stress around the hole has practically returned to the value of the undisturbed plate at about $2.5r_L$ from the hole centre [NACA740]. This clearance keeps the stringer welds out of the region of raised stress around the hole.

Both rules contain the factor 2.5, but the first is applied to the diameter d and the second to the radius r_L ; the two distances therefore differ by a factor of two.

Position of the minimum (derived here). The sum of the moments is proportional to the mean curvature of the plate, $M_r + M_\varphi = -K(1 + \nu)(w'' + w'/r)$ (Eq. (23)). With Eqs. (3) and (7):

$$M_r + M_\varphi = \frac{pa^2}{16} [2(3 + \nu) - 4(1 + \nu)\rho^2 - 4\eta] = 0$$

gives Eq. (16). Inside this radius the plate is curved like a bowl; outside it the clamping moment at the edge dominates and the hole-edge stress rises again.

In-plane load. In a closed vessel the wall carries the axial force $n_x = pR_m/2$, so the effective radial load is $p_w = p - \nu n_x/R_m = p(1 - \nu/2)$ [W] and [W]. The free expansion follows from Eq. (4.2-65) without the bending term, $\delta = p_w R_m^2/D_y$, with $D_y = Et_T$. The plate prevents this expansion. The upper bound of the edge force follows from the same wall solution with free rotation, $u = C e^{-\beta x} \cos(\beta x)$, which gives a radial stiffness of $2\beta^3 B_M$ per wall half (derived here). Taking both halves as fully restrained gives $Q \leq 4\beta^3 B_M \delta$, Eq. (17).

Linearity. Membrane action becomes noticeable from $w/s > 1/7$ and dominates above $w > 3s$ [W]. With $w_0/s_P = 3.3$ the unstiffened plate lies in the membrane range; this is why only the distribution over r_0 is used. The stiffened plate will be designed to stay in the linear range.

2 Stiffening of the Bulkhead

2.1 Purpose and Design Basis

The 8 mm plate of Sec. 1 cannot carry the pressure difference alone. As a simply supported plate it would deflect by 26.4 mm (Eq. (14)), and its bending stress would lie far above the yield strength. This section dimensions the stringers that are welded onto the plate, so that the plate keeps its shape and the stresses stay within the limit at the smallest possible mass.

Load case. Both compartments operate at 7 bar, so nominally the bulkhead sees no pressure difference. The design load is the emergency case of Sec. 1: one compartment loses its pressure, and $\Delta p = 0.7 \text{ MPa}$ acts on the plate, in either direction. In this case the bulkhead has to prevent a breach between LOX and LCH₄. Plastic deformation is accepted: after such an event the tank may be unusable, but the two propellants must not come into contact.

Design basis.

- Material 1.4404 (316L). The design is made for the operating temperature, with the yield strength at -170°C as limit, $\sigma_{lim} = R_{p0.2} = 500 \text{ MPa}$, without safety factor. The higher strength of 316L at cryogenic temperature is used on purpose; $\Delta p = 0.7 \text{ MPa}$ is therefore only allowed with the bulkhead cold. At room temperature the allowed pressure difference is lower (Sec. 2.3.15).
- Elastic design: the stresses at the web tips and in the plate stay at or below σ_{lim} . Local peaks at the joints may exceed it, as plastic deformation is accepted in the emergency case.
- Deflection $w_{max} \leq 4 \text{ mm}$ ($= 0.5 s_P$); 2 mm and 1 mm are studied as variants.
- Stringers from 2 mm sheet, mirrored on both sides of the plate, $10 \text{ mm} \leq h \leq 40 \text{ mm}$, joined by laser welds. The welds cannot be inspected; the calculation uses the weld factor $z = 0.7$.
- The opening of Sec. 1 at $r_0 = 190 \text{ mm}$ keeps a zone free of stringers: no stringer centre line closer than $2.5 r_L = 31.25 \text{ mm}$ to the hole centre (Eqs. (24), (25)).

Maturity and validation. The bulkhead is an experimental part; the target is TRL 4 to 5. The stress-strain curve of the actual sheet has not been measured, so the margin up to rupture is not quantified. The analysis shows that the design meets the elastic criteria at operating temperature. The pressure tests validate it: water at room temperature at the reduced pressure of Sec. 2.4.6, LN₂ at operating pressure, later the real propellants. The validation test at full pressure difference lasts only a few seconds.

Approach. Hand models come first: the radial stringers alone as an upper bound, then the rings with the force method and a smeared model of the plate, the buckling of the webs, and the hub with the joints. Python scripts evaluate all layout variants. A plate model with beam elements checks the hand models, and an Ansys model of the complete tank checks the final design (Sec. 2.5).

2.2 Result

The bulkhead is stiffened by 16 radial stringers, a hub ring and three rings, all with webs of $h = 30 \text{ mm}$ on both sides of the plate (Table 5, Fig. 20). Each stringer consists of one web on each side of the plate; the 16 radial stringers are therefore 32 radial webs. Table 6 compares the models for this layout, and Table 7 gives the allowed pressure difference.

Design path.

1. Radial stringers alone need $h \geq 30.3 \text{ mm}$ with $n_R \leq 19$. Above $h \approx 17 \text{ mm}$ the stress governs, not the deflection (Sec. 2.5.1).

Table 5: Final geometry of the stiffened bulkhead. Radii of the rings refer to the centre line of the web; all webs are mirrored on both sides of the plate.

Part	Quantity	Value
Plate	material	1.4404 (316L)
	radius at the support a	273 mm
	thickness s_P	8 mm
Tank wall	mean radius R_m , thickness t_T	274 mm, 2 mm
Opening	diameter d	25 mm
	position	$r_0 = 190$ mm, $\theta = 0^\circ$ (field middle)
Radial stringers	number n_R	16 (one web per side, 32 webs)
	angular position	$\theta = 11.25^\circ + k \cdot 22.5^\circ$, $k = 0 \dots 15$
	start	outer face of the hub, $r = 67$ mm
	end	45° chamfer, $r = 241.5 \dots 268.5$ mm ($h = 30 \rightarrow 3$ mm, middle at $r_e = 255$ mm)
Hub ring	web thickness t_s , height h per side	2 mm, 30 mm
	crossings with the rings	slot 2 mm wide, raised bridge over the ring
	outer radius, wall thickness t_h	67 mm, 5 mm ($r = 62 \dots 67$ mm)
	mean radius of the wall	64.5 mm (strip length, mass)
	radius in the force method R_h	66 mm (Sec. 2.4.5)
Rings	height h per side	30 mm
	construction	rolled strip $405 \times 30 \times 5$ mm, one seam
	radii R_j	91 mm, 116 mm, 158.75 mm
	web thickness t_s , height h per side	2 mm, 30 mm
All webs	construction	bent strip welded to a cylinder, in the slots
Overall height	drain holes at the plate	half holes 3 mm
Welds	$2h + s_P$	68 mm
Mass	type	laser, through the web ($a_w = 2$ mm), $z = 0.7$
	stringers, plate	6.10 kg, 15.0 kg

Table 6: Final design at $\Delta p = 0.7$ MPa and operating temperature. Hand: force method, plate neglected, simply supported (Sec. 2.4.5). ODE: smeared plate model, simply supported or with the edge restraint of the tank wall (Eq. (58), Sec. 2.5.6). Ansys: complete tank (Sec. 2.5.7). Stresses in MPa; –: not evaluated in this model.

Quantity	hand	ODE, simply supp.	ODE, tank wall	Ansys
w_{max} [mm]	1.80	1.63	1.55	1.5
radial stringers, web tip	486.7	441.2	419.5	–
hub ($t_h = 5$ mm)	266.1	255.3	245.5	–
ring 91 mm	263.0	250.6	240.9	–
ring 116 mm	279.5	264.6	254.2	–
ring 158.75 mm	313.0	293.2	280.9	–
web tip in the chamfer	–	199.2	157.5	no peak
hub wall between the webs (local)	298.5	–	–	–
plate, largest global value	–	68.4	65.1	–
local peaks at the ring fillets	–	–	–	480–510

Table 7: Allowed pressure difference (Sec. 2.4.6).

Condition	Δp_{max} [MPa]	expected w_{max} [mm]
bulkhead at operating temperature (LN2 test, operation)	0.70	1.5
bulkhead at room temperature (water test)	0.47	1.0

2. Rings only pay when the web height is limited. For $h \leq 30$ mm the lightest layout is $n_R = 16$ with one ring at 158.75 mm, the inner edge of the zone around the opening (Sec. 2.5.2).
3. The plate clamps the webs against buckling. The full $\sigma_{lim} = 500$ MPa is available up to $h = 30.4$ mm; $h = 30$ mm is chosen (Sec. 2.5.3).
4. At the centre the radial stringers end at a hub; two extra rings at 91 and 116 mm relieve it (Sec. 2.5.4).
5. The beam–plate FE model confirms the hand model for stringers and rings, but it cannot resolve the hub junction (Sec. 2.5.5).
6. In the Ansys model of the complete tank the radial webs bend the 2 mm hub wall in the plane of the plate between the webs (826 MPa; hand estimate 875 MPa). The hub wall is therefore made 5 mm thick (estimate 299 MPa). The straight stringer ends give a peak at the cut; they end with a 45° chamfer instead (Sec. 2.5.7).

Results.

- All models stay below σ_{lim} outside the joints. The hand model gives the largest values (487 MPa in the radial stringers just outside the main ring); the ODE with the plate and Ansys give less. The deflection is 1.5 mm to 1.8 mm, well below 4 mm. ODE with the edge restraint of the tank wall and Ansys agree within 0.05 mm.
- Ansys shows the largest stresses, 480 MPa to 510 MPa, at the fillets of the rings next to the welds. The fillets are modelled coarsely; the real value depends on the weld (a laser weld without filler gives hardly any fillet). These are local peaks at about σ_{lim} and are accepted for the emergency case.
- With the 5 mm hub the welds of hub and rings stay below $\approx \sigma_{lim} = 350$ MPa in the hand model. The welds of the slots, where the rings run through the radial webs, carry the stress of the radial stringers (up to 487 MPa at the main ring) and exceed this value (Table 10). They are not part of the barrier between the propellants; the pressure tests validate them.
- The water test at room temperature is limited to 0.47 MPa; at operating temperature the bulk-head takes 0.7 MPa. The stringers weigh 6.10 kg.

Limits of the analysis.

- The stress–strain curve of the sheet has not been measured. The margin up to rupture is not quantified, and the allowed pressure at room temperature relies on the temperature dependence reported in [Li2023].
- The welds are not inspected; the pressure tests validate them.
- In the Ansys model the parts are bonded to each other, so the slots at the ring crossings (Sec. 3) act as fully welded; the welds themselves are not modelled.
- Web buckling of the final layout is checked with the hand model only (Sec. 2.5.3), not with an eigenvalue analysis of the Ansys model. In Ansys the ring webs tend to deflect sideways.

2.3 Analytical Model

This section lists the formulas used in the calculation: first the description of the stringers and their coupling with the plate, then the models of the single design steps. Their derivation and the sources are collected in Sec. 2.6.

2.3.1 Layout and Design Variables

The stiffened plate is described as a flat surface with lines on it. Each line is the centre line of a stringer (Fig. 3). Where two lines cross, the stringers are welded together and act as one joint.

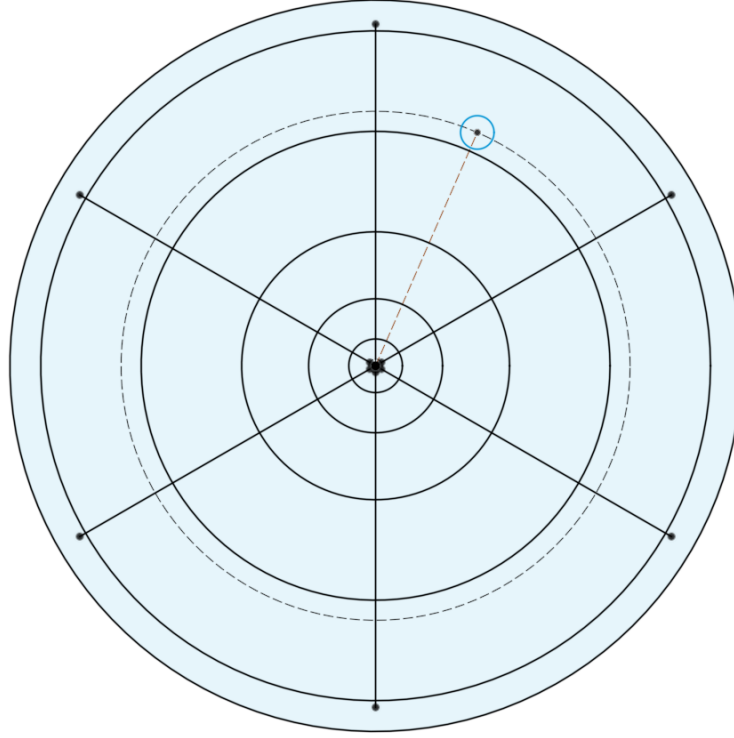


Figure 3: Layout sketch (example with six radial stringers and five rings). Solid lines: stringer centre lines; dashed circle: $r_0 = 190$ mm; marked point: 25 mm opening.

- **Radial stringers:** n_R straight stringers, equally spaced at the angle $2\pi/n_R$, from $r_i = 5$ mm (clearance circle at the centre) to $r_e = 255$ mm. In the final layout they start at a hub ring, in the model at $R_h = 66$ mm (Sec. 2.3.12).
- **Ring stringers:** n_C concentric rings at the radii R_j , $j = 1 \dots n_C$. The radii are set individually (Sec. 2.3.10).
- **Cross-section:** all stringers are identical. Each is a web of thickness $t_s = 2$ mm and height h on both sides of the plate (mirrored), with $10 \text{ mm} \leq h \leq 40 \text{ mm}$.
- **Design variables:** h , n_R , n_C and the radii R_j .

Constraints from the position of the opening (Sec. 1, $r_0 = 190$ mm, hole radius $r_L = d/2 = 12.5$ mm). The distance from the hole centre to the centre line of every stringer is at least $2.5 r_L = 31.25$ mm (Sec. 2.6.1):

$$\text{rings:} \quad |R_j - r_0| \geq 2.5 r_L \quad \Rightarrow \quad R_j \leq 158.75 \text{ mm or } R_j \geq 221.25 \text{ mm}, \quad (24)$$

$$\text{radial stringers:} \quad \frac{2\pi r_0}{n_R} \geq 5 r_L \quad \Rightarrow \quad n_R \leq 19. \quad (25)$$

The angular position of the radial stringers is free, so the opening is placed inside a field between two of them. The clearance $2.5 r_L$ is not the distance $2.5 d$ between hole centre and plate edge of Eq. (15); both rules are compared in Sec. 1.6.5.

2.3.2 Load Path

The pressure acts on the plate. The plate fields between the stringers carry it as a line load to the stringers along their centre lines. The radial stringers and rings carry it by bending to the outer region,

where the plate passes it as shear into the tank wall. Between $r_e = 255$ mm and the wall at $a = 273$ mm there is only the 8 mm plate, which transmits little moment; for the stringer grid this region acts close to a hinge. The load can act in both directions. Because the webs are mirrored, both directions give the same stresses with opposite sign, and in each case one web half is in compression.

2.3.3 Stringer Cross-Section

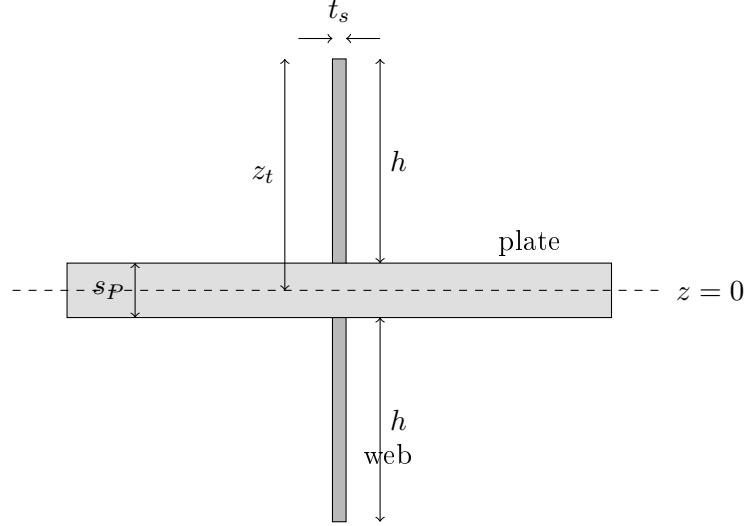


Figure 4: Cross-section of a stringer: mirrored webs, plate at the mid-plane $z = 0$.

Because the webs are mirrored, the plate mid-plane is the neutral plane of plate and stringers alike. Plate and stringer stiffnesses therefore simply add, and no effective plate width is needed for the stringer. The stringer is described by its webs alone, with respect to the plate mid-plane (Fig. 4):

$$I_w = \frac{t_s}{12} \left[(2h + s_P)^3 - s_P^3 \right], \quad (26)$$

$$z_t = \frac{s_P}{2} + h, \quad W_t = \frac{I_w}{z_t}, \quad (27)$$

$$S_w = t_s h \frac{s_P + h}{2}. \quad (28)$$

z_t is the distance of the web tip from the mid-plane, W_t the section modulus at the tip and S_w the first moment of one web about the mid-plane (needed for the weld). The torsional stiffness of the thin open webs is neglected.

The stiffener ratio compares one stringer with the plate stiffness K (Eq. (2)):

$$b_{eq} = \frac{E I_w}{K}, \quad (29)$$

i.e. the width of plain plate that has the same bending stiffness as one stringer.

Table 8: Cross-section values ($t_s = 2$ mm, $s_P = 8$ mm, $E = 200\,000$ MPa, $K = 9.377 \times 10^6$ N · mm).

h [mm]	I_w [mm ⁴]	z_t [mm]	W_t [mm ³]	$E I_w$ [N · mm ²]	b_{eq} [mm]	S_w [mm ³]
10	3573.33	14	255.24	7.147×10^8	76.2	180.0
20	18346.67	24	764.44	3.669×10^9	391.3	560.0
30	52320.00	34	1538.82	1.046×10^{10}	1115.9	1140.0
40	113493.33	44	2579.39	2.270×10^{10}	2420.6	1920.0

A single stringer with $h = 10$ mm is as stiff as 76 mm of plate, with $h = 40$ mm as stiff as 2.4 m of plate. Within the allowed range, I_w changes by a factor of 32.

2.3.4 Radial Stringers

A radial stringer is a straight beam along r , welded to the plate along its whole length. It deflects and rotates with the plate, so its curvature is the radial curvature of the plate:

$$\kappa_R = -\frac{\partial^2 w}{\partial r^2}, \quad (30)$$

$$M_R = E I_w \kappa_R, \quad \sigma_R(z) = \frac{M_R}{I_w} z, \quad \sigma_{R,t} = \frac{M_R}{W_t} = E \kappa_R z_t, \quad (31)$$

$$\frac{dQ_R}{dr} = -q_R, \quad \frac{dM_R}{dr} = Q_R, \quad (32)$$

$$T_R = \frac{Q_R S_w}{I_w}. \quad (33)$$

M_R is the bending moment and Q_R the shear force of the stringer, $\sigma_{R,t}$ the stress at the web tip and T_R the shear flow in the weld between one web and the plate (force per unit length). The line load q_R is the load passed on by the two adjacent plate fields. If the plate itself and the rings did not carry anything, each radial stringer would take the full sector of angle $2\pi/n_R$ (upper bound):

$$q_R(r) \leq p \frac{2\pi r}{n_R}. \quad (34)$$

At the centre all radial stringers are connected through the plate and the innermost ring, which balance their moments across the clearance circle (for even n_R , two opposite stringers act as one continuous beam).

Smeared over the circumference, the radial stringers add the radial bending stiffness

$$S_R = \frac{n_R E I_w}{2\pi} \quad \text{per radian}, \quad D_R(r) = \frac{S_R}{r} \quad \text{per unit circumferential length}, \quad (35)$$

which decreases outwards because the stringers move apart. They add nothing in the circumferential direction.

2.3.5 Ring Stringers

A ring of radius R is welded to the plate along its whole circumference. Its web follows the deflection and the slope of the plate, so its curvature is the circumferential curvature of the plate at $r = R$:

$$\kappa_C = -\left(\frac{1}{R} \frac{\partial w}{\partial r} + \frac{1}{R^2} \frac{\partial^2 w}{\partial \theta^2}\right)_{r=R}, \quad (36)$$

$$M_C = E I_w \kappa_C, \quad \sigma_{C,t} = \frac{M_C}{W_t} = E \kappa_C z_t. \quad (37)$$

Under rotationally symmetric load ($\partial/\partial\theta = 0$) only the first term remains. With the plate slope $\varphi = \partial w/\partial r$ at R , the ring then acts on the plate as a rotational spring:

$$\kappa_C = -\frac{\varphi}{R}, \quad m_C = k_C \varphi, \quad k_C = \frac{E I_w}{R^2}, \quad (38)$$

with the edge moment m_C per unit circumferential length. Smeared over its tributary radial width l_C , a ring adds the circumferential bending stiffness

$$D_C = \frac{E I_w}{l_C} \quad (39)$$

and nothing in the radial direction. Between two radial stringers a ring segment also carries the line load of the adjacent plate fields as a curved beam with the span $2\pi R/n_R$.

2.3.6 Plate between the Stringers

The plate carries its own share with the stiffness K in both directions (Eq. (23)). From the global curvature its surface stress is

$$\sigma_{P,r} = \frac{E}{1-\nu^2} (\kappa_r + \nu \kappa_\varphi) \frac{s_P}{2}, \quad \sigma_{P,\varphi} = \frac{E}{1-\nu^2} (\kappa_\varphi + \nu \kappa_r) \frac{s_P}{2}, \quad (40)$$

with $\kappa_r = -\partial^2 w / \partial r^2$ and $\kappa_\varphi = -\frac{1}{r} \partial w / \partial r - \frac{1}{r^2} \partial^2 w / \partial \theta^2$. In addition, each field bends locally between its stringers. A field is bounded by two radial stringers (width $b_\theta = 2\pi r / n_R - t_s$) and two rings (length l in radial direction); its local bending is treated in Sec. 2.3.10.

2.3.7 Coupled System

Plate and stringers share one deflection field $w(r, \theta)$. The total potential energy of the stiffened plate is

$$\Pi = \frac{1}{2} \iint_A \boldsymbol{\kappa}^T \mathbf{D}_b \boldsymbol{\kappa} \, dA + \sum_{\text{radial}} \frac{1}{2} \int_{r_i}^{r_e} E I_w \kappa_R^2 \, dr + \sum_{j=1}^{n_C} \frac{1}{2} \oint E I_w \kappa_C^2 R_j \, d\theta - \iint_A p w \, dA, \quad (41)$$

with the plate curvatures $\boldsymbol{\kappa}$ and $\mathbf{D}_b$ from Sec. 1.5. This is the basis of the FE model, where the stringers enter as beam elements along the mesh lines.

For hand calculations the stringers are smeared. For rotationally symmetric load the moments per unit length are then

$$m_r = K (\kappa_r + \nu \kappa_\varphi) + D_R(r) \kappa_r, \quad m_\varphi = K (\nu \kappa_r + \kappa_\varphi) + D_C \kappa_\varphi, \quad (42)$$

$$\frac{d}{dr} (r m_r) - m_\varphi = -\frac{p r^2}{2}, \quad (43)$$

with $\kappa_r = -w''$ and $\kappa_\varphi = -w'/r$. Eq. (43) is the equilibrium of a plate element; it is solved for the stringer layouts in Secs. 2.3.9 and 2.3.10.

2.3.8 Mass

With the density $\rho = 8.0 \text{ kg/dm}^3$, the mass of all stringers (overlaps at the crossings neglected) is

$$m_s = \rho 2h \left[t_s \left(n_R (r_e - r_i) + 2\pi \sum_{j=1}^{n_C} R_j \right) + t_h 2\pi R_{h,m} \right], \quad (44)$$

with the inner end r_i of the radial stringers ($r_i = R_h$ with a hub) and the mean radius $R_{h,m}$ of the hub wall. One metre of stringer weighs 0.32 kg for $h = 10 \text{ mm}$ and 1.28 kg for $h = 40 \text{ mm}$. For comparison, the plate itself weighs 15.0 kg.

2.3.9 Radial Stringers Alone

Bounding model: the plate and the rings carry nothing; the radial stringers are smeared over the circumference (stiffness S_R per radian, Eq. (35)) and simply supported at $r = a$. The stringer stiffness is taken over the full radius; the centre and the plain plate ring between r_e and a are neglected.

Moment per radian, stringer moment at the centre, deflection and shear force:

$$m(r) = \frac{p}{6} (a^3 - r^3), \quad M_{R,max} = \frac{2\pi}{n_R} \frac{p a^3}{6}, \quad (45)$$

$$w(r) = \frac{p}{120 S_R} [10 a^3 (a^2 - r^2) - (a^5 - r^5)], \quad w_{max} = \frac{3 p a^5}{40 S_R}, \quad (46)$$

$$Q_R(r) = \frac{2\pi}{n_R} \frac{p r^2}{2}. \quad (47)$$

The stress at the web tip in the centre, $\sigma_t = M_{R,max}/W_t$, is tied to the deflection independently of n_R :

$$\sigma_t = \frac{20 E z_t}{9 a^2} w_{max}. \quad (48)$$

Required number of radial stringers for a deflection limit w_{lim} and a stress limit σ_{lim} :

$$n_{R,w} = \frac{2\pi \cdot 3 p a^5}{40 E I_w w_{lim}}, \quad n_{R,\sigma} = \frac{2\pi p a^3}{6 W_t \sigma_{lim}}, \quad n_R \geq \max(n_{R,w}, n_{R,\sigma}). \quad (49)$$

2.3.10 Rings and Plate

- **Hand model (force method):** the radial stringers of Sec. 2.3.9 are the statically determinate base system. Each ring acts on it as a line moment X_j per radian at R_j , equal to its bending moment M_C (Eq. (37)). The X_j are the redundants and follow from the compatibility of ring and stringer rotation at R_j . The plate is neglected (conservative).
- **Plate model (ODE):** Eq. (43) with the plate, the radial stringers up to r_e , plate only between r_e and a , and the rings as discrete jumps of the moment; solved numerically.
- **Plate fields:** plate strip clamped at two neighbouring radial stringers.
- Simply supported at $r = a$. For the final layout the ODE is also solved with the edge restraint by the tank wall (Sec. 1.3.3).
- Ring positions within the bands of Eq. (24), on a 5 mm grid, from $R = 20$ mm up to r_e . For welding access two rings are at least 25 mm apart, and at each ring the field between two radial stringers is at least 25 mm wide, $2\pi R_j/n_R \geq 25$ mm.
- Layout search: all ring layouts that fit these rules are evaluated with the hand model, from no ring up to the maximum of eight rings, each with $n_R = 3 \dots 19$.

Compatibility of the n_C rings with the base system:

$$\alpha_{j0} + \sum_{k=1}^{n_C} \alpha_{jk} X_k = -c_j X_j, \quad j = 1 \dots n_C, \quad (50)$$

$$\alpha_{j0} = -\frac{p R_j}{6 S_R} \left(a^3 - \frac{R_j^3}{4} \right), \quad \alpha_{jk} = \frac{\min(R_j, R_k)}{S_R}, \quad c_j = \frac{R_j}{E I_w}.$$

For a single ring at R :

$$X = \frac{2\pi p (a^3 - R^3/4)}{6 (n_R + 2\pi)}. \quad (51)$$

Moment per radian, stringer moment and stresses:

$$M(r) = \frac{p}{6} (a^3 - r^3) - \sum_{R_j > r} X_j, \quad M_R = \frac{2\pi}{n_R} M, \quad (52)$$

$$\sigma_{R,t} = \frac{M_{R,max}}{W_t}, \quad M_{R,max} = \frac{2\pi}{n_R} \max \left\{ M(0), M(R_j^+) \right\}, \quad \sigma_{C,t} = \frac{X_j}{W_t}. \quad (53)$$

Deflection at the centre:

$$w_{max} = \frac{3 p a^5}{40 S_R} - \frac{1}{S_R} \sum_{j=1}^{n_C} X_j R_j \left(a - \frac{R_j}{2} \right). \quad (54)$$

The X_j depend only on n_R and R_j , not on h . Each layout therefore gives a required W_t and I_w , and the smallest h follows from Eqs. (26) and (27).

Plate model (ODE), with $\varphi = w'$ and the moment per radian $M = r m_r$ of plate and stringers (Eqs. (42), (43)):

$$\varphi' = -\frac{M + K\nu\varphi}{rK + S_R(r)}, \quad M' = -K\nu\varphi' - \frac{K\varphi}{r} - \frac{pr^2}{2}, \quad M(R_j^+) = M(R_j^-) - \frac{EI_w\varphi(R_j)}{R_j}, \quad (55)$$

with $S_R(r) = S_R$ for $r < r_e$, $S_R(r) = 0$ for $r > r_e$, and $\varphi(0) = 0$, $M(a) = 0$, $w(a) = 0$.

Plate stress at the stringer ends, plate field at radius r (width b):

$$\sigma_{P,e} = \frac{p(a^3 - r_e^3)}{r_e s_P^2}, \quad (56)$$

$$b = \frac{2\pi r}{n_R} - t_s, \quad w_f = \frac{pb^4}{384K},$$

$$\sigma_f = \frac{pb^2}{2s_P^2} \text{ (at the stringer)}, \quad \sigma_{f,m} = \frac{pb^2}{4s_P^2} \text{ (mid-field)}. \quad (57)$$

At the opening the mid-field stress is added to the global plate stress (Eq. (40)), $\sigma_\varphi + \sigma_{f,m}$ and $\sigma_r + \nu\sigma_{f,m}$, and inserted into Eq. (13).

For the final layout the edge restraint by the tank wall replaces the simple support. With the rotational stiffness k_M of the wall per unit length (Eq. (9)) the edge conditions become

$$M(a) = a k_M \varphi(a), \quad w(a) = 0. \quad (58)$$

2.3.11 Web Buckling

- The compressed web half of a radial stringer or ring is a long plate strip of height h and thickness t_s with a free tip edge. The stress is linear over the height, from $\sigma_t s_P/(2z_t)$ at the root to σ_t at the tip (Eq. (31)). The support of the free edge at crossings with other stringers is neglected (conservative).
- When the web buckles, its root rotates and the plate has to rotate with it along the root line. The plate therefore acts as a rotational spring. Only 10 % of the spring stiffness of the two adjacent plate halves is used (conservative; covers the finite field width and the weld).
- The buckling coefficient follows from an energy (Ritz) solution: sine along the web, polynomial over the height. The model is checked against the exact values of TM4 Tab. 5.1.
- Plasticity is taken into account through an effective modulus from secant and tangent modulus. The stress-strain curve at operating temperature is not known; it is described by Ramberg-Osgood with $R_{p0.2} = 500$ MPa, and the lowest buckling stress for the exponents $m = 4 \dots 20$ is used. The upper limit is $R_{p0.2}$.
- The same limit applies to radial stringers and rings; the layout search is the one of Sec. 2.3.10 (all ring layouts, force method, plate neglected).

Buckling stress at the web tip and rotational spring of the plate:

$$\sigma_{cr} = k_\sigma E_w \left(\frac{t_s}{h} \right)^2, \quad k_\theta = f \frac{4\pi K}{L}, \quad f = 0.1, \quad (59)$$

with the half-wave length L of the buckle. Ritz solution with $v(x, z) = \sin(\pi x/L) \sum_i c_i (z/h)^i$ (z from the root):

$$U_w = \frac{D_w}{2} \iint [(v_{,xx} + v_{,zz})^2 - 2(1 - \nu)(v_{,xx}v_{,zz} - v_{,xz}^2)] dx dz, \quad U_\theta = \frac{1}{2} \int k_\theta v_{,z}^2|_{z=0} dx,$$

$$W = \frac{1}{2} \iint \sigma(z) t_s v_{,x}^2 dx dz, \quad \delta(U_w + U_\theta - W) = 0, \quad (60)$$

$$(\mathbf{K} - \sigma_t \mathbf{G}) \mathbf{c} = \mathbf{0}, \quad k_\sigma = \min \frac{\sigma_t}{E} \left(\frac{h}{t_s} \right)^2,$$

with $D_w = Et_s^3/(12(1 - \nu^2))$. Effective modulus and allowable stress:

$$\varepsilon = \frac{\sigma}{E} + 0.002 \left(\frac{\sigma}{R_{p0.2}} \right)^m, \quad E_s = \frac{\sigma}{\varepsilon}, \quad E_t = \frac{d\sigma}{d\varepsilon}, \quad (61)$$

$$E_w = 0.43 E_s + 0.29 E_b, \quad E_b = \sqrt{E_s^2 + 3 E_s E_t}, \quad (62)$$

$$\sigma_{lim}(h) = \min \left\{ R_{p0.2}, \min_m \sigma_{cr}(h, m) \right\}, \quad M_{max} \leq W_t(h) \sigma_{lim}(h). \quad (63)$$

σ_{cr} appears on both sides of Eq. (59) through E_w and is found by iteration.

2.3.12 Centre Hub and Joints

- A small central tube does not work: 16 webs of 2 mm need 32 mm of circumference (tube with 10 mm diameter: 31.4 mm), and a ring at which all n_R stringers end must take $n_R/(2\pi) = 2.55$ times the moment of one stringer. The radial stringers therefore end at a hub ring of radius $R_h = 66$ mm (field width $2\pi R_h/n_R = 26$ mm ≥ 25 mm), made like the rings. R_h is the radius at which the force method connects the radial stringers to the hub; for the 2 mm hub it is the centre line of the wall ($r = 65 \dots 67$ mm). Inside the hub there is only the plate.
- Force method of Sec. 2.3.10 with the base system starting at the hub: the hub carries the whole moment per radian at R_h and acts as a rotational spring; the rings are the redundants as before. The plate inside the hub is a simply supported disk whose edge follows the hub rotation.
- Extra rings between hub and main ring relieve the centre. With the spacing of 25 mm at most two fit, at 91 mm and 116 mm.
- Assumed construction of the joints: radial webs continuous from the hub to r_e ; each ring bent from strip, closed by one seam and interrupted by the radial webs, its ends welded to them (crossing); web ends welded to the hub; all welds are laser welds through the 2 mm web ($a_w = t_s$). Allowable stress $z \sigma_{lim}$ with the weld factor $z = 1$ (full penetration, tested) or $z = 0.7$ (rule for welds without testing). In the final construction the rings are continuous instead and run through slots in the radial webs (Sec. 3).
- Stringer ends at r_e cut straight; the force of the web is assumed to pass into the plate over the length h . The final layout uses a chamfer (Sec. 2.3.14).

Compatibility with hub ($c_h = R_h/(EI_h)$, $I_h = I_w t_h/t_s$, $M_0(r) = p(a^3 - r^3)/6$):

$$\sum_j X_j \left[c_h + \frac{\min(R_j, R_k) - R_h}{S_R} \right] + c_k X_k = c_h M_0(R_h) + \frac{1}{S_R} \int_{R_h}^{R_k} M_0 dr. \quad (64)$$

Hub moment, stresses and deflection:

$$M_h = M_0(R_h) - \sum_j X_j, \quad \sigma_H = \frac{M_h}{W_t} \frac{t_s}{t_h}, \quad \sigma_{end} = \frac{2\pi}{n_R} \frac{M_h}{W_t}, \quad (65)$$

$$w_{max} = - \int_{R_h}^a \varphi dr + |\varphi_h| \frac{R_h}{2} + \frac{5 + \nu}{1 + \nu} \frac{p R_h^4}{64K}, \quad \varphi(r) = \varphi_h - \frac{1}{S_R} \int_{R_h}^r M dr, \quad \varphi_h = -c_h M_h. \quad (66)$$

Welds (a_w load-bearing thickness):

$$\sigma_v = \sqrt{\sigma_{\perp}^2 + 3\tau_{\parallel}^2} \leq z \sigma_{lim}, \quad (67)$$

$$\text{web-plate: } \tau_{\parallel} = \frac{Q_R S_w}{I_w a_w}, \quad \sigma_{\perp} = \frac{2\pi}{n_R} \frac{p r}{2 a_w}, \quad Q_R = \frac{2\pi}{n_R} \frac{p r^2}{2}, \quad (68)$$

$$\text{ring crossing, seam: } \sigma_{\perp} = \frac{X_j}{W_t}, \quad \Delta M_R = 2X_j \sin \frac{\pi}{n_R} \text{ (into the radial stringer)}, \quad (69)$$

$$\text{stringer end: } \tau_{\parallel} = \frac{1}{a_w} \left(\frac{Q_R S_w}{I_w} + \frac{F_w}{h} \right),$$

$$F_w = \sigma_e t_s \frac{z_t^2 - s_P^2/4}{2 z_t}, \quad \sigma_e = \frac{2\pi}{n_R} \frac{M_0(r_e)}{W_t}. \quad (70)$$

The web ends at the hub carry $\sigma_{\perp} = \sigma_{end}$, the hub seam $\sigma_{\perp} = \sigma_H$. With continuous rings that run through slots in the radial webs, the ring seam carries $\sigma_{\perp} = X_j/W_t$, and the welds of the slot carry the stress of the radial stringer just outside the ring:

$$\sigma_{\perp} = \frac{2\pi}{n_R} \frac{M(R_j^+)}{W_t}. \quad (71)$$

2.3.13 Hub Wall between the Webs

The radial webs end at the hub. The force in the web tips is passed into the hub wall at n_R points; between them the wall bends in the plane of the plate.

Hub wall between the webs, as a ring loaded by the n_R web forces F at the web ends ($\alpha = \pi/n_R$, plate support neglected):

$$F = \sigma_{end} t_s \frac{z_t^2 - s_P^2/4}{2 z_t}, \quad N = \frac{F}{2 \tan \alpha},$$

$$M_L = \frac{F R_h}{2} \left(\frac{1}{\alpha} - \cot \alpha \right), \quad M_M = -\frac{F R_h}{2} \left(\frac{1}{\sin \alpha} - \frac{1}{\alpha} \right), \quad \sigma = \frac{N}{h t_h} \pm \frac{6 M}{h t_h^2}, \quad (72)$$

with M_L at the web and M_M in the middle between two webs (bending in the plane of the plate).

2.3.14 Stringer Ends

In the first layouts the radial webs end with a straight cut at r_e , and the web force passes into the plate at the end (Eq. (70)). The final layout ends with a 45° chamfer of the web tip, with its middle at r_e , from the full height h down to $h_{end} = 3$ mm:

$$h(r) = \begin{cases} h, & r \leq r_e - \Delta, \\ h - (r - r_e + \Delta), & r_e - \Delta < r \leq r_e + \Delta, \end{cases} \quad \Delta = \frac{h - h_{end}}{2} = 13.5 \text{ mm}. \quad (73)$$

In the ODE, $S_R(r)$ follows from $I_w(h(r))$, and the stress at the web tip from $z_t(r) = s_P/2 + h(r)$.

2.3.15 Allowed Pressure at Room Temperature

All stresses and the deflection are linear in the pressure. The design reaches σ_{lim} at $p = 0.7$ MPa; the pressure with the same utilisation at room temperature is

$$p_{RT} = p \frac{R_{p0.2}(\text{RT})}{\sigma_{lim}}, \quad R_{p0.2}(\text{RT}) = \sigma_{lim} \frac{R_L(297 \text{ K})}{R_L(T_{op})}, \quad w_{RT} = w_{max} \frac{p_{RT}}{p}, \quad (74)$$

where $R_L(T)$ is the yield strength of 316L measured at the temperature T [Li2023] and $T_{op} = 103$ K (-170°C). Only the ratio of the measured values is used; the sheet is assumed to follow the same temperature dependence.

2.3.16 Design Criteria and Failure Modes

Table 9 collects the criteria of the layout with the equations of this section and the sections in which they are evaluated.

Table 9: Criteria for the stringer layout and the sections in which they are treated.

Criterion / failure mode	Quantity	Sec.
Deflection	$w_{max} \leq 4 \text{ mm } (= 0.5 s_P)$; variants 2 mm, 1 mm	2.5.1, 2.5.2
Stress at the web tip	$\sigma_{R,t}, \sigma_{C,t} \leq \sigma_{lim}$, Eqs. (31), (37)	2.5.1, 2.5.2
Stress in the plate	σ_P , Eq. (40)	2.5.2
Local bending of the plate fields	field size b_θ, l , Eq. (57)	2.5.2
Buckling of the compressed web half	free web edge under $\sigma_{R,t}, \sigma_{C,t}$	2.5.3
Welds	web-plate, crossings, seams and slots, Eqs. (67)–(71)	2.5.4, 2.4.5
Centre junction	hub, Eqs. (64)–(66), (72)	2.5.4, 2.4.5
Ends of the radial stringers	web force into the plate, Eq. (70); chamfer geometry, Eq. (73)	2.5.4, 2.5.7
Discrete stringers, joints	FE models	2.5.5, 2.5.7
Test at room temperature	p_{RT} , Eq. (74)	2.4.6

2.4 Worked Calculation

The calculation chain is shown with the examples used during the design (Secs. 2.4.1–2.4.3) and for the hub and the final layout (Secs. 2.4.4–2.4.6). The scripts of Sec. 2.5 reproduce all values.

2.4.1 Example: Radial Stringers Alone

With $h = 40 \text{ mm}$ and $n_R = 12$:

$$\begin{aligned}
 S_R &= \frac{12 \cdot 200000 \cdot 113493.33}{2\pi} = 4.335 \times 10^{10} \text{ N} \cdot \text{mm}^2 \\
 w_{max} &= \frac{3 \cdot 0.7 \cdot 273^5}{40 \cdot 4.335 \cdot 10^{10}} = 1.836 \text{ mm} \\
 M_{R,max} &= \frac{2\pi}{12} \cdot \frac{0.7 \cdot 273^3}{6} = 1.243 \times 10^6 \text{ N} \cdot \text{mm}, \quad \sigma_t = \frac{1.243 \cdot 10^6}{2579.39} = 481.9 \text{ MPa} \\
 \text{check: } \sigma_t &= \frac{20 \cdot 200000 \cdot 44}{9 \cdot 273^2} \cdot 1.836 = 481.9 \text{ MPa} \\
 Q_R(r_e) &= \frac{2\pi}{12} \cdot \frac{0.7 \cdot 255^2}{2} = 11\,916 \text{ N}, \quad m_R = 8.0 \cdot 10^{-6} \cdot 2 \cdot 2 \cdot 40 \cdot 12 \cdot 250 = 3.84 \text{ kg}
 \end{aligned}$$

2.4.2 Example: One Ring

With $h = 40 \text{ mm}$, $n_R = 12$ and one ring at $R = 221.25 \text{ mm}$.

With $S_R = 4.335 \times 10^{10} \text{ N} \cdot \text{mm}^2$ and $E I_w = 2.270 \times 10^{10} \text{ N} \cdot \text{mm}^2$ from Sec. 2.4.1:

$$\begin{aligned}\alpha_{10} &= -\frac{0.7 \cdot 221.25}{6 \cdot 4.335 \cdot 10^{10}} \left(273^3 - \frac{221.25^3}{4} \right) = -1.0503 \times 10^{-2} \\ \alpha_{11} + c_1 &= \frac{221.25}{4.335 \cdot 10^{10}} + \frac{221.25}{2.270 \cdot 10^{10}} = 1.4851 \times 10^{-8} / (\text{N} \cdot \text{mm}) \\ X &= \frac{1.0503 \cdot 10^{-2}}{1.4851 \cdot 10^{-8}} = 7.072 \times 10^5 \text{ N} \cdot \text{mm} \\ X &= \frac{2\pi \cdot 0.7 \cdot 1.7639 \cdot 10^7}{6 \cdot 18.283} = 7.072 \times 10^5 \text{ N} \cdot \text{mm} \quad (\text{check with Eq. (51)}) \\ M(0) &= \frac{0.7 \cdot 273^3}{6} - 7.072 \cdot 10^5 = 1.667 \times 10^6 \text{ N} \cdot \text{mm} \\ M(R^+) &= \frac{0.7 (273^3 - 221.25^3)}{6} = 1.110 \times 10^6 \text{ N} \cdot \text{mm} < M(0) \\ \sigma_{R,t} &= \frac{2\pi}{12} \cdot \frac{1.667 \cdot 10^6}{2579.39} = 338.3 \text{ MPa}, \quad \sigma_{C,t} = \frac{7.072 \cdot 10^5}{2579.39} = 274.2 \text{ MPa} \\ w_{max} &= 1.836 - \frac{7.072 \cdot 10^5 \cdot 221.25 \cdot (273 - 110.625)}{4.335 \cdot 10^{10}} = 1.836 - 0.586 = 1.250 \text{ mm} \\ m_s &= 8.0 \cdot 10^{-6} \cdot 2 \cdot 2 \cdot 40 (12 \cdot 250 + 2\pi \cdot 221.25) = 5.62 \text{ kg}\end{aligned}$$

Compared with the radial stringers alone (482 MPa, 1.836 mm) the ring lowers the stringer stress by 30 % and the deflection by 32 %. The ODE with the plate gives $w_{max} = 1.207 \text{ mm}$, $\sigma_{R,t} = 322.8 \text{ MPa}$, $\sigma_{C,t} = 260.9 \text{ MPa}$.

2.4.3 Example: Web Buckling

With $h = 40 \text{ mm}$:

$$\begin{aligned}D_w &= \frac{200000 \cdot 2^3}{12 (1 - 0.3^2)} = 1.465 \times 10^5 \text{ N} \cdot \text{mm}, \quad K = 9.377 \times 10^6 \text{ N} \cdot \text{mm} \\ k_\sigma &= 1.373 \text{ at } L = 1.71 h = 68 \text{ mm} \quad (\text{root simply supported } 0.499, \text{ clamped } 1.422) \\ k_\theta &= 0.1 \cdot \frac{4\pi \cdot 9.377 \cdot 10^6}{68.2} = 1.73 \times 10^5 \text{ N}, \quad \frac{k_\theta h}{D_w} = 47 \\ \sigma_{cr,el} &= 1.373 \cdot 200000 \cdot \left(\frac{2}{40} \right)^2 = 686.5 \text{ MPa} > R_{p0.2} \Rightarrow \text{plastic} \\ \varepsilon &= \frac{417.6}{200000} + 0.002 \cdot 0.835^4 = 0.003061 \quad (m = 4, \sigma = 417.6 \text{ MPa}) \\ E_s &= \frac{417.6}{0.003061} = 136\,425 \text{ MPa}, \quad E_t = \left[\frac{1}{200000} + \frac{0.002 \cdot 4}{500} 0.835^3 \right]^{-1} = 69\,832 \text{ MPa} \\ E_b &= \sqrt{136425^2 + 3 \cdot 136425 \cdot 69832} = 217\,238 \text{ MPa} \\ E_w &= 0.43 \cdot 136425 + 0.29 \cdot 217238 = 121\,662 \text{ MPa} \\ \text{check: } \sigma_{cr} &= 1.373 \cdot 121662 \cdot 0.05^2 = 417.6 \text{ MPa} = \sigma_{lim}(40 \text{ mm})\end{aligned}$$

2.4.4 Hub and Joints with a 2 mm Hub

Hub at 66 mm with $t_h = 2$ mm, rings at 91, 116 and 158.75 mm:

$$X = (5.647, 5.402, 5.544) \cdot 10^5 \text{ N} \cdot \text{mm} \quad (\text{Eq. (64), script})$$

$$M_h = \frac{0.7(273^3 - 66^3)}{6} - 16.592 \cdot 10^5 = 23.402 \cdot 10^5 - 16.592 \cdot 10^5 = 6.810 \times 10^5 \text{ N} \cdot \text{mm}$$

$$\sigma_H = \frac{6.810 \cdot 10^5}{1538.8} = 442.5 \text{ MPa}, \quad \sigma_{end} = \frac{2\pi}{16} \cdot 442.5 = 173.8 \text{ MPa}$$

$$\text{rings: } \sigma_{\perp} = 367.0, 351.0, 360.2 \text{ MPa}$$

$$\Delta M_R = 2 \cdot 5.544 \cdot 10^5 \cdot \sin \frac{\pi}{16} = 2.163 \times 10^5 \text{ N} \cdot \text{mm} \quad (\text{at } 158.75 \text{ mm})$$

$$Q_R(r_e) = \frac{2\pi}{16} \cdot \frac{0.7 \cdot 255^2}{2} = 8937 \text{ N}, \quad \tau_{\parallel} = \frac{8937 \cdot 1140}{52320 \cdot 2} = 97.4 \text{ MPa}$$

$$\sigma_{\perp} = \frac{2\pi}{16} \cdot \frac{0.7 \cdot 255}{2 \cdot 2} = 17.5 \text{ MPa}$$

$$\sigma_v = \sqrt{17.5^2 + 3 \cdot 97.4^2} = 169.6 \text{ MPa} \quad (\text{web-plate weld at } r_e)$$

$$\sigma_e = \frac{2\pi}{16} \cdot \frac{4.393 \cdot 10^5}{1538.8} = 112.1 \text{ MPa}, \quad F_w = 112.1 \cdot 2 \cdot \frac{34^2 - 16}{68} = 3758 \text{ N}$$

$$\tau_{\parallel} = \frac{194.7 + 3758/30}{2} = 160.0 \text{ MPa}$$

$$\sigma_v = \sqrt{17.5^2 + 3 \cdot 160.0^2} = 277.7 \text{ MPa} \quad (\text{stringer end})$$

Hub wall (Eq. (72)) with $\sigma_{end} = 173.8$ MPa, $\alpha = \pi/16$:

$$F = 173.8 \cdot 2 \cdot \frac{34^2 - 16}{68} = 5827 \text{ N}, \quad \frac{N}{h t_h} = \frac{5827 \cdot 2.514}{30 \cdot 2} = 244.1 \text{ MPa}$$

$$M_L = \frac{5827 \cdot 66}{2} \cdot 0.0656 = 1.262 \times 10^4 \text{ N} \cdot \text{mm}, \quad \frac{6 M_L}{h t_h^2} = \frac{6 \cdot 1.262 \cdot 10^4}{30 \cdot 4} = 630.9 \text{ MPa}$$

At the web this gives $244 + 631 = 875$ MPa (estimate, see Sec. 2.6).

2.4.5 Final Hub Wall ($t_h = 5$ mm)

The force method keeps the hub radius $R_h = 66$ mm of the 2 mm hub. The 5 mm wall lies at $r = 62 \dots 67$ mm: the radial webs start at its outer face ($r = 67$ mm), and the strip length and the mass use its mean radius (64.5 mm). $R_h = 66$ mm lies between both. With R_h between 64.5 and 67 mm the results of the final layout change by at most 1 % (S2g_FinalDesign.py with R_h varied).

With $I_h = I_w t_h/t_s = 2.5 \cdot 52320 = 130800 \text{ mm}^4$ and the other values as above, Eq. (64) gives:

$$c_h = \frac{66}{200000 \cdot 130800} = 2.523 \times 10^{-9} / (\text{N} \cdot \text{mm})$$

$$X = (4.048, 4.302, 4.817) \cdot 10^5 \text{ N} \cdot \text{mm} \quad (\text{script})$$

$$M_h = 23.402 \cdot 10^5 - 13.167 \cdot 10^5 = 1.0236 \times 10^6 \text{ N} \cdot \text{mm}$$

$$\sigma_H = \frac{1.0236 \cdot 10^6}{1538.8} \cdot \frac{2}{5} = 266.1 \text{ MPa}, \quad \sigma_{end} = \frac{2\pi}{16} \cdot \frac{1.0236 \cdot 10^6}{1538.8} = 261.2 \text{ MPa}$$

$$\text{rings: } \sigma_{C,t} = \frac{X_j}{W_t} = 263.0, 279.5, 313.0 \text{ MPa}, \quad w_{max} = 1.80 \text{ mm}$$

$$F = 261.2 \cdot 2 \cdot \frac{34^2 - 16}{68} = 8758 \text{ N}, \quad \frac{N}{h t_h} = \frac{8758 \cdot 2.514}{30 \cdot 5} = 146.8 \text{ MPa}$$

$$M_L = \frac{8758 \cdot 66}{2} \cdot 0.0656 = 1.897 \times 10^4 \text{ N} \cdot \text{mm}, \quad \frac{6 M_L}{h t_h^2} = \frac{6 \cdot 1.897 \cdot 10^4}{30 \cdot 25} = 151.7 \text{ MPa}$$

At the web the hub wall carries $147 + 152 = 299$ MPa instead of 875 MPa. The stiffer hub attracts more moment: M_h rises by 50 %, the rings are relieved by 13 to 28 %, and the web ends at the hub carry more. The radial stringers outside the main ring are not affected (486.7 MPa). The hub wall weighs $8.0 \cdot 10^{-6} \cdot 60 \cdot 5 \cdot 2\pi \cdot 64.5 = 0.97$ kg instead of 0.40 kg; with the chamfered ends the stringers weigh 6.10 kg.

In the final construction the rings are continuous and run through slots in the radial webs (Sec. 3). The ring moment then passes only through the ring seam (J2), while the moment of the radial stringer crosses each ring through the welds of the slot (J6, Eq. (71)):

$$M(R_j^+) = (13.740, 17.100, 19.070) \cdot 10^5 \text{ N} \cdot \text{mm} \quad (R_j = 91, 116, 158.75 \text{ mm})$$

$$\sigma_{\perp} = \frac{2\pi}{16} \cdot \frac{M(R_j^+)}{1538.8} = 350.6, 436.4, 486.7 \text{ MPa}$$

Table 10 gives all joints.

Table 10: Joints of the final layout ($\sigma_{lim} = 500$ MPa, $a_w = 2$ mm, hub seam through the 5 mm wall).
*Reference value for a straight end; the final stringer ends are chamfered (J5).

Joint	Load	σ_v [MPa]	$z = 1$	$z = 0.7$
J1 web-plate weld (r_e , straight end)*	shear flow, load introduction	169.6	0.34	0.48
J2 ring 91 mm: seam	ring moment	263.0	0.53	0.75
J2 ring 116 mm: seam	ring moment	279.5	0.56	0.80
J2 ring 158.75 mm: seam	ring moment	313.0	0.63	0.89
J3 web ends at the hub	stringer moment, shear	261.2	0.52	0.75
J4 hub seam	hub moment	266.1	0.53	0.76
J5 stringer end, chamfer	Ansys without peak	not evaluated by hand		
J6 slot at ring 91 mm	stringer moment	350.6	0.70	1.00
J6 slot at ring 116 mm	stringer moment	436.4	0.87	1.25
J6 slot at ring 158.75 mm	stringer moment	486.7	0.97	1.39

2.4.6 Pressure at Room Temperature

Measured yield strength of 316L [Li2023]: 990 MPa at 50 K, 860 MPa at 173 K and 660 MPa at 297 K. Linear interpolation to $T_{op} = 103$ K and Eq. (74):

$$R_L(103 \text{ K}) = 990 + \frac{103 - 50}{173 - 50} (860 - 990) = 934 \text{ MPa}$$

$$R_{p0.2}(\text{RT}) = 500 \cdot \frac{660}{934} = 353 \text{ MPa}, \quad p_{RT} = 0.7 \cdot \frac{353}{500} = 0.495 \text{ MPa}$$

The interpolation is an assumption. With the measured values on both sides of T_{op} the result lies between $0.7 \cdot 660/990 = 0.467$ MPa and $0.7 \cdot 660/860 = 0.537$ MPa. The lower bound is used:

$$p_{RT} = 0.47 \text{ MPa}, \quad w_{RT} = 1.5 \cdot \frac{0.47}{0.7} = 1.0 \text{ mm} \quad (\text{from Ansys}).$$

The absolute value at 297 K is not used. The material of [Li2023] is much stronger at low temperature than the design value (860 MPa at 173 K against 500 MPa); with $R_{p0.2}(\text{RT}) = 660$ MPa the water test would even exceed the operating pressure. The deflection at the centre can be measured during the water test and compared with w_{RT} .

2.5 Simulations

The Python scripts evaluate the models of Sec. 2.3 for all variants; each prints its values with three decimals and draws the figures of this section. The beam-plate FE model of Sec. 2.5.5 checks the hand models, and the Ansys model of Sec. 2.5.7 checks the final design.

2.5.1 Radial Stringers over the Web Height

S2b_RadialStringers.py evaluates Eq. (49) over h and prints all values (Fig. 5, Table 11).

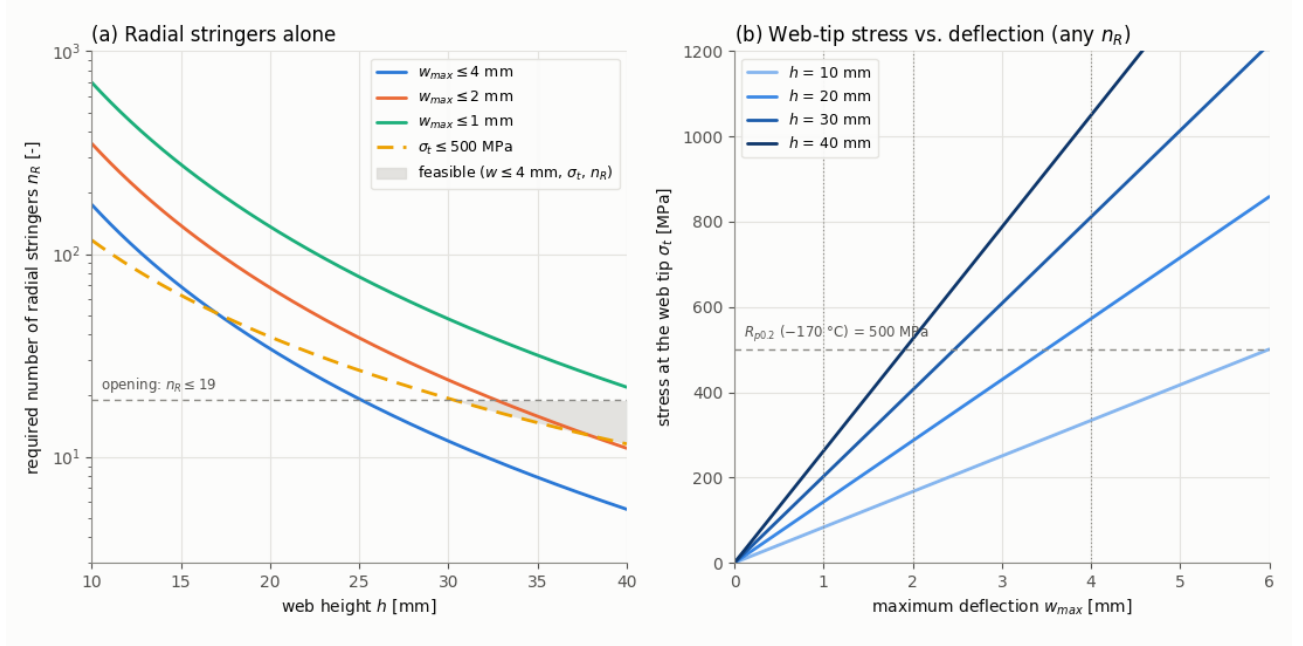


Figure 5: Radial stringers alone. (a) Required number n_R over the web height for the deflection limits and the stress limit; grey: feasible for $w_{max} \leq 4$ mm, $\sigma_t \leq 500$ MPa and $n_R \leq 19$. (b) Stress at the web tip over the deflection, Eq. (48).

Table 11: Required number of radial stringers (radial stringers alone, continuous values).

h [mm]	$w \leq 4$ mm	$w \leq 2$ mm	$w \leq 1$ mm	$\sigma_t \leq 500$ MPa
10	175.0	350.0	699.9	116.9
20	34.1	68.2	136.3	39.0
30	12.0	23.9	47.8	19.4
40	5.5	11.0	22.0	11.6

Results.

- Above $h \approx 17$ mm the stress governs, not the deflection: at $\sigma_t = 500$ MPa the deflection is only 3.5 mm ($h = 20$), 2.5 mm ($h = 30$) and 1.9 mm ($h = 40$).
- With $n_R \leq 19$ the radial stringers alone need $h \geq 30.3$ mm ($w \leq 4$ mm and $\sigma_t \leq 500$ MPa), $h \geq 32.7$ mm for $w \leq 2$ mm; $w \leq 1$ mm is not reachable ($h \geq 42.2$ mm).
- Example $h = 40$ mm, $n_R = 12$: $w_{max} = 1.84$ mm, $\sigma_t = 482$ MPa, 3.84 kg.
- This is an upper bound. The plate and the rings, both neglected here, carry part of the load; their effect is quantified in Sec. 2.5.2.

2.5.2 Ring Layouts

S2c_RingsPlate.py evaluates the example, solves the ODE (Fig. 6) and scans the ring positions for every n_R and $n_C \leq 3$; for each combination the positions with the smallest required h are kept (Fig. 7). The ODE is verified against three limits: plate only gives $w_0 = 26.413$ mm (Eq. (14)), stringers only ($K \rightarrow 0$, $r_e = a$) give Eq. (46), and with rings the force method; all three agree to the printed digits.

S2c2_RingStudy.py evaluates all 71 204 ring layouts with 0 to 8 rings, each with $n_R = 3 \dots 19$ (1.21×10^6 combinations, of which 4.56×10^5 meet the field width). For every layout the required h follows from Eqs. (50)–(54). The lightest layout for each maximum web height h_{max} forms the front in Fig. 8 and Table 12; the layouts along it are drawn in Fig. 9. The script also shows the required h over n_R and n_C (map and 3D surface) and a map over the radii of two rings; it reproduces the values of S2c_RingsPlate.py for $n_C \leq 3$.

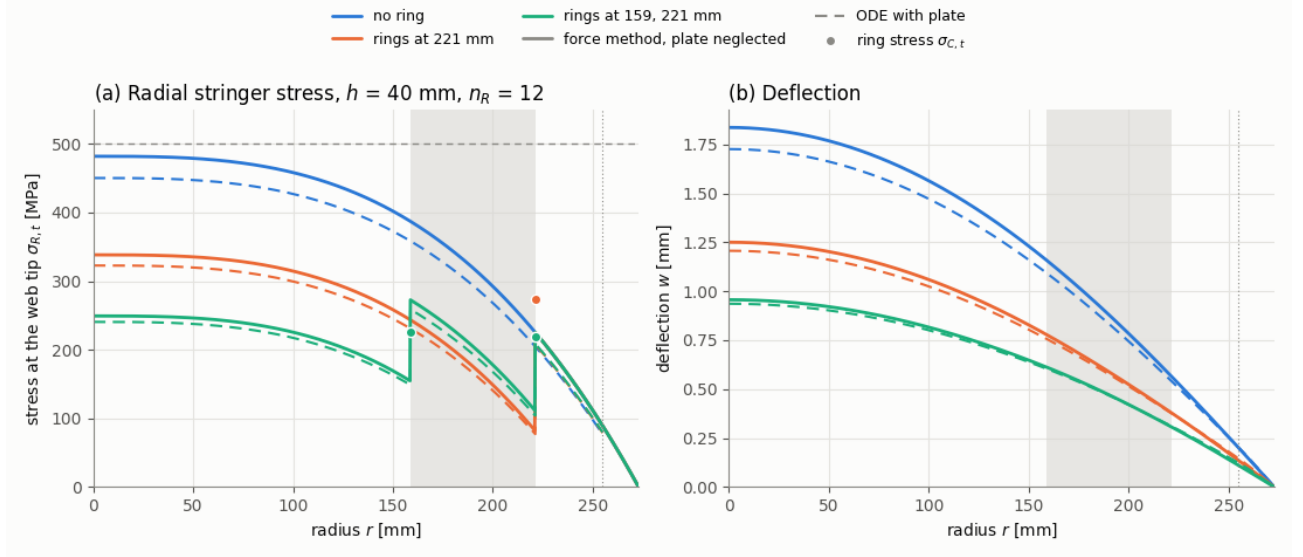


Figure 6: Example $h = 40$ mm, $n_R = 12$ with no, one and two rings. Solid: force method (plate neglected); dashed: ODE with plate; dots: ring stress. (a) Stress at the web tip of the radial stringers, (b) deflection. Grey band: opening at r_0 .

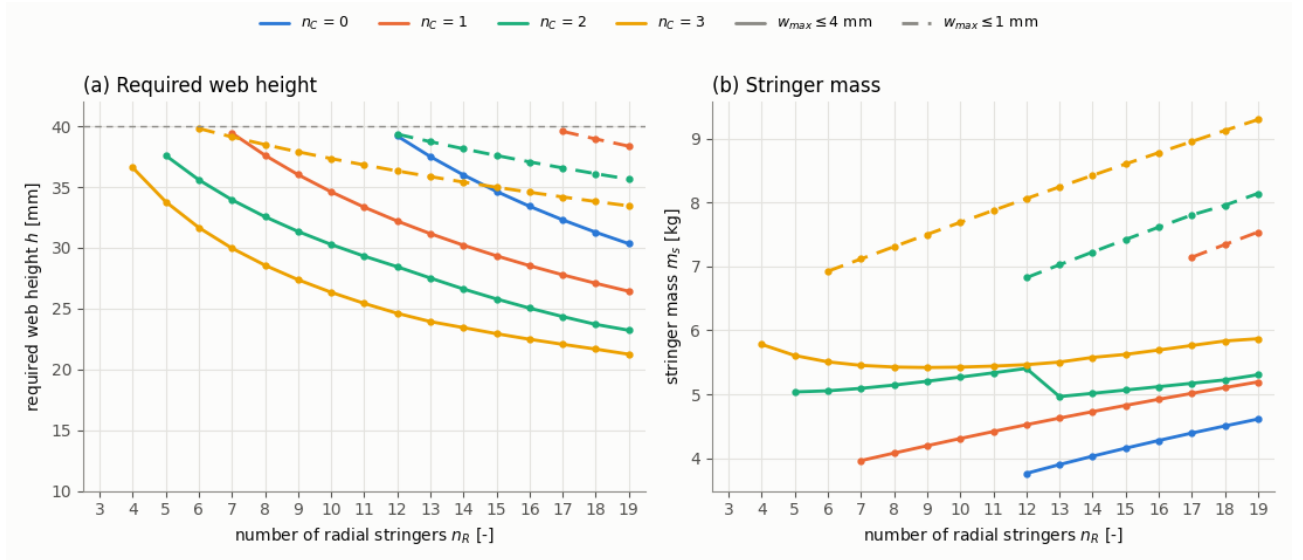


Figure 7: Smallest required web height (a) over n_R for $n_C = 0 \dots 3$ rings at the positions that give the smallest h , and the stringer mass of these layouts (b); solid: $w_{max} \leq 4$ mm, dashed: $w_{max} \leq 1$ mm; both with $\sigma_{R,t}, \sigma_{C,t} \leq 500$ MPa, plate neglected. The kink of $n_C = 2$ in (b) is a change of the best positions.

Results.

- A ring takes a moment X that is independent of h and lowers the stringer moment inside its radius. For the smallest h the rings lie at the edges of the band around the opening: first $R = 221.25$ mm, then 158.75 mm, then 246.25 mm ($= 221.25 + 25$ mm). The ideal position of

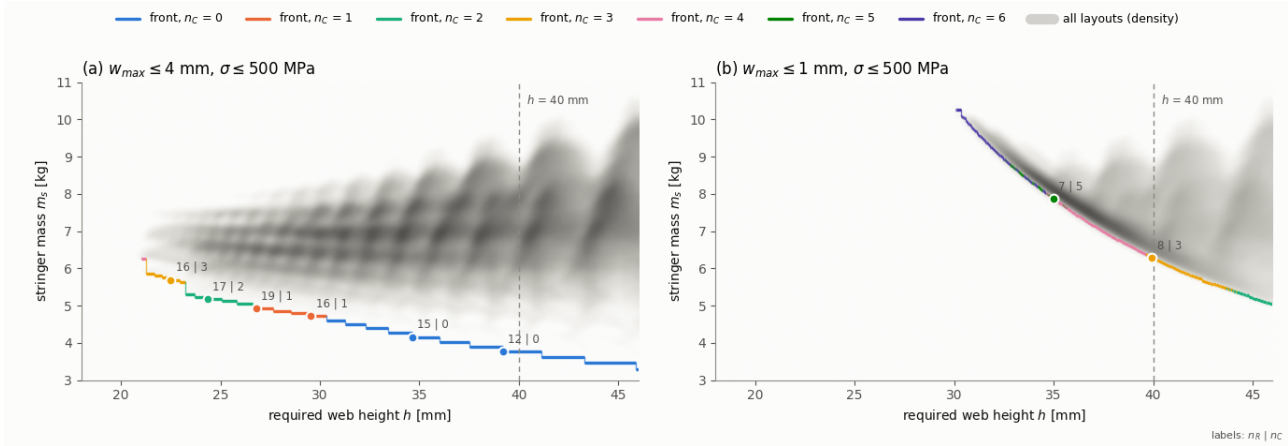


Figure 8: All layouts in the plane of required web height and stringer mass (grey: density) and the lightest layout for each maximum web height (front, coloured by the number of rings); labels $n_R | n_C$. (a) $w_{max} \leq 4$ mm, (b) $w_{max} \leq 1$ mm; both with $\sigma \leq 500$ MPa, plate neglected.

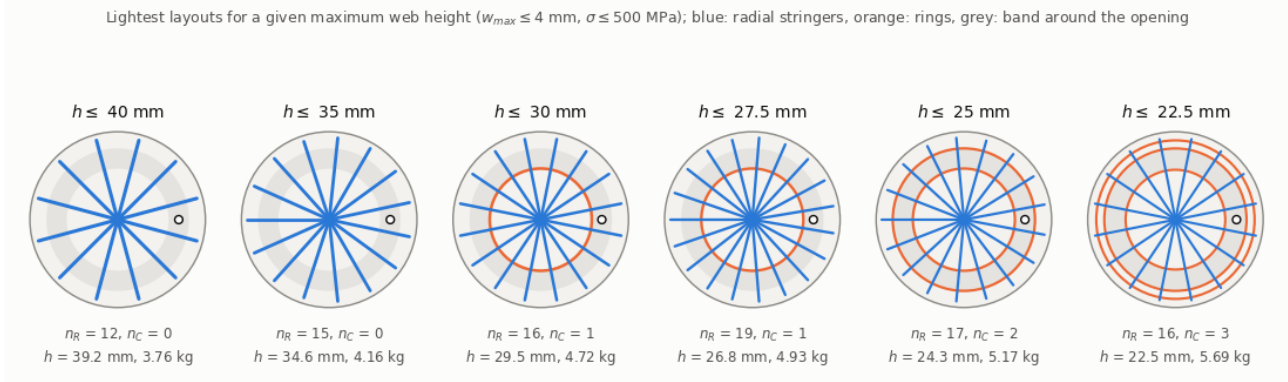


Figure 9: Lightest layouts along the front for $w_{max} \leq 4$ mm (Table 12). Blue: radial stringers, orange: rings, grey: band around the opening.

Table 12: Lightest layout for a given maximum web height h_{max} (all ring layouts, force method, plate neglected, $\sigma_{R,t}, \sigma_{C,t} \leq 500$ MPa).

w_{lim} [mm]	h_{max} [mm]	n_R	n_C	R_j [mm]	h [mm]	m_s [kg]
4	40	12	0	—	39.2	3.76
4	35	15	0	—	34.6	4.16
4	30	16	1	158.75	29.5	4.72
4	27.5	19	1	158.75	26.8	4.93
4	25	17	2	158.75, 221.25	24.3	5.17
4	22.5	16	3	155, 221.25, 246.25	22.5	5.69
4	min.	19	4	105, 158.75, 221.25, 246.25	21.0	6.29
2	40	12	0	—	39.2	3.76
2	35	12	1	158.75	34.7	4.44
2	30	11	3	50, 158.75, 221.25	30.0	5.22
2	27.5	14	4	60, 85, 158.75, 221.25	27.4	5.97
2	25	14	5	75, 125, 158.75, 221.25, 246.25	25.0	6.95
2	min.	19	6	80, 105, 130, 158.75, 221.25, 246.25	23.1	7.87
1	40	8	3	85, 158.75, 221.25	39.9	6.29
1	35	7	5	85, 130, 158.75, 221.25, 246.25	35.0	7.88
1	min.	19	6	80, 105, 130, 158.75, 221.25, 246.25	30.1	10.26

a single ring ($n_R = 12$: about 186 mm, where $M(0) = M(R^+)$) lies inside the band. For the smallest mass a single ring moves to 158.75 mm, because it is shorter there.

- As long as h may go up to 40 mm, rings do not pay: every ring lowers h , but not enough to make up for its length. The lightest layout is $n_R = 12$ without ring ($h = 39.2$ mm, 3.76 kg, $w_{max} = 1.94$ mm). Rings enter the lightest layout only when the web height is limited (Table 12, Fig. 9): one ring at 158.75 mm below $h_{max} \approx 30$ mm, two and three rings below about 25 mm.
- More than four rings do not lower h further. The moment just outside the band ($r = R_{in}^+ = 158.75$ mm) can only be lowered by rings at $R \geq 221.25$ mm, and only two of them fit there. The smallest possible web height is 21.0 mm ($n_R = 19$, four rings).
- With $w \leq 2$ mm the deflection governs below about $h = 38$ mm (without rings exactly 37.9 mm, Eq. (48)); inner rings mainly help the deflection, so the lightest layouts take more rings. $w \leq 1$ mm needs $h \geq 30.1$ mm, rings and at least 6.29 kg.
- The plate lowers the deflection by 6 % to 9 % and the stringer stress by 7 % to 12 %; the hand model is conservative, and the radial stringers alone are confirmed as an upper bound.
- Plate: the global stress is largest at the stringer ends, 157 MPa (hand: 161.5 MPa), almost independent of the layout. Field at r_e for $n_R \geq 7$: $\sigma_f \leq 282$ MPa and $w_f \leq 0.52$ mm; opening: $\sigma_L \leq 240$ MPa. The plate does not govern.
- Buckling of the compressed web half ($h/t_s \approx 20$) follows in Sec. 2.5.3, the local bending of the ring segments between the radial stringers and the discrete stringer positions in Secs. 2.5.5 and 2.5.7.

2.5.3 Web Buckling and Layout

S2d_WebBuckling.py checks the Ritz solution against TM4 Tab. 5.1 (largest deviation 0.6 %), computes k_σ and σ_{lim} over h (Fig. 10) and repeats the layout search of Sec. 2.5.2 with $\sigma_{lim}(h)$ (Fig. 11, Fig. 12, Table 13).

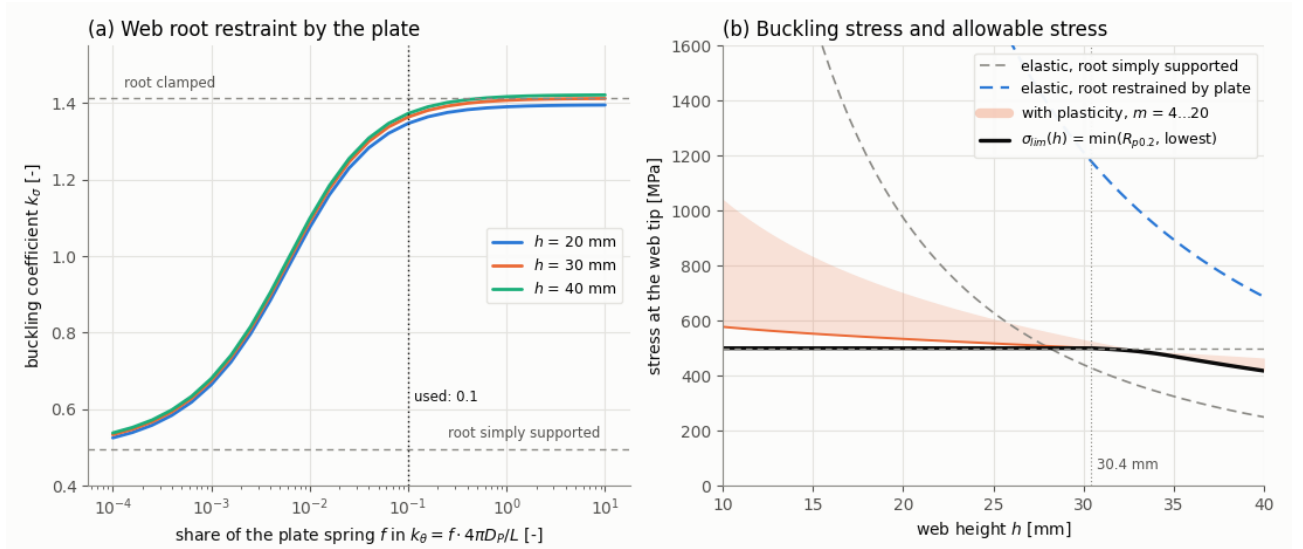


Figure 10: (a) Buckling coefficient over the share f of the plate spring; (b) buckling stress at the web tip and allowable stress $\sigma_{lim}(h)$ (band: plastic buckling stress for $m = 4 \dots 20$).

Results.

- The 8 mm plate clamps the 2 mm web practically rigidly: 10 % of its spring stiffness already gives 97 % of the clamped buckling coefficient ($k_\sigma = 1.31 \dots 1.37$ for $h = 10 \dots 40$ mm). With a simply supported root it would be only $k_\sigma \approx 0.5$.

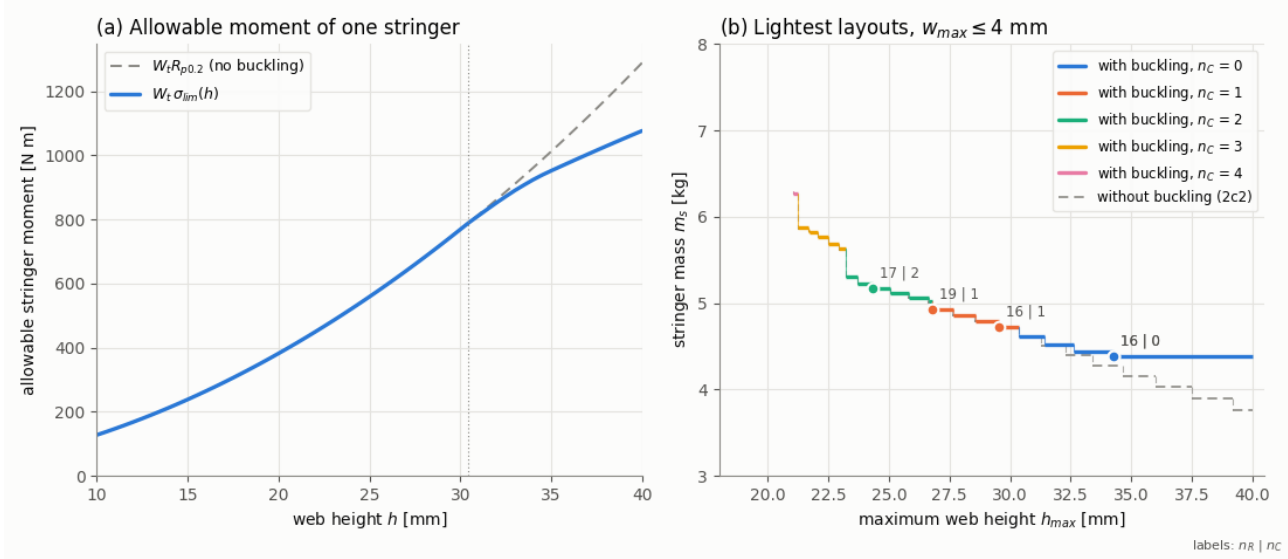


Figure 11: (a) Allowable moment of one stringer with and without web buckling; (b) lightest layout for each maximum web height with web buckling (coloured by n_C) and without (Sec. 2.5.2, dashed), $w_{max} \leq 4$ mm; labels $n_R | n_C$.

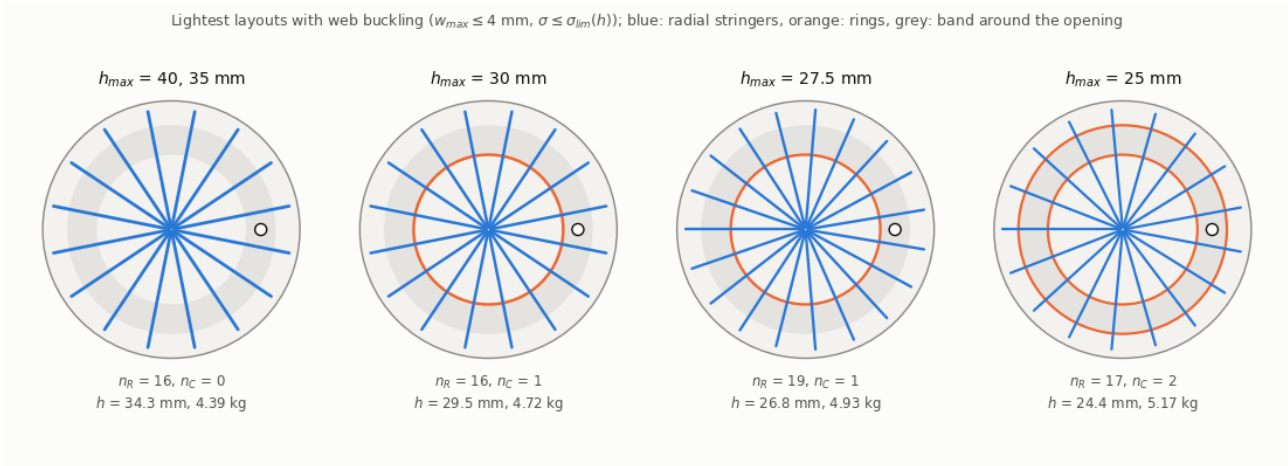


Figure 12: Lightest layouts with web buckling, $w_{max} \leq 4$ mm.

Table 13: Lightest layout for a given maximum web height with $\sigma \leq \sigma_{lim}(h)$ (all ring layouts, force method, plate neglected).

w_{lim} [mm]	h_{max} [mm]	n_R	n_C	R_j [mm]	h [mm]	m_s [kg]
4	40, 35	16	0	—	34.3	4.39
4	30	16	1	158.75	29.5	4.72
4	27.5	19	1	158.75	26.8	4.93
4	25	17	2	158.75, 221.25	24.4	5.17
2	40	15	0	—	36.6	4.40
2	35	16	0	—	34.9	4.46
2	30	11	3	50, 158.75, 221.25	30.0	5.23
2	27.5	14	4	60, 85, 158.75, 221.25	27.4	5.97
2	25	14	5	75, 125, 158.75, 221.25, 246.25	25.0	6.95
1	40	8	3	85, 158.75, 221.25	39.9	6.29

- Elastically the buckling stress stays above $R_{p0.2}$ up to $h \approx 47$ mm, but it then lies in the plastic range. The full $R_{p0.2} = 500$ MPa is available up to $h = 30.4$ mm; above, σ_{lim} drops to 470 MPa at 35 mm and 418 MPa at 40 mm.
- The allowable moment of one stringer still grows with h , but more slowly: at $h = 40$ mm it is 1.077×10^6 N · mm instead of 1.290×10^6 N · mm without buckling.
- The lightest layout without height limit becomes $n_R = 16$ without rings, $h = 34.3$ mm, 4.39 kg for $w \leq 4$ mm (without buckling: $n_R = 12$, 3.76 kg); for $w \leq 2$ mm $n_R = 15$, $h = 36.6$ mm, 4.40 kg. Below $h \approx 30$ mm nothing changes compared with Sec. 2.5.2.
- Chosen for the following steps: $n_R = 16$, one ring at 158.75 mm, $h = 30$ mm (from $h_{max} = 30$ mm, rounded). The full $R_{p0.2}$ is available; the largest stress at the web tip is 487 MPa (just outside the ring), $w_{max} = 2.25$ mm, $m_s = 4.80$ kg without the centre. The ring is kept on purpose; the centre follows in Sec. 2.5.4. (Without rings, $n_R = 16$ with $h = 35$ mm would be 4.48 kg.)
- The joints and the FE checks follow in Secs. 2.5.4–2.5.7.

2.5.4 Hub Variants and Joints

S2e_HubJoints.py solves Eq. (64) for the variants of Table 14, with the hub wall chosen so that $\sigma_H \leq \sigma_{lim}$, scans the radius of one extra ring and computes the weld stresses of the preliminary layout with 2 mm hub (Fig. 13, Fig. 14, Table 15).

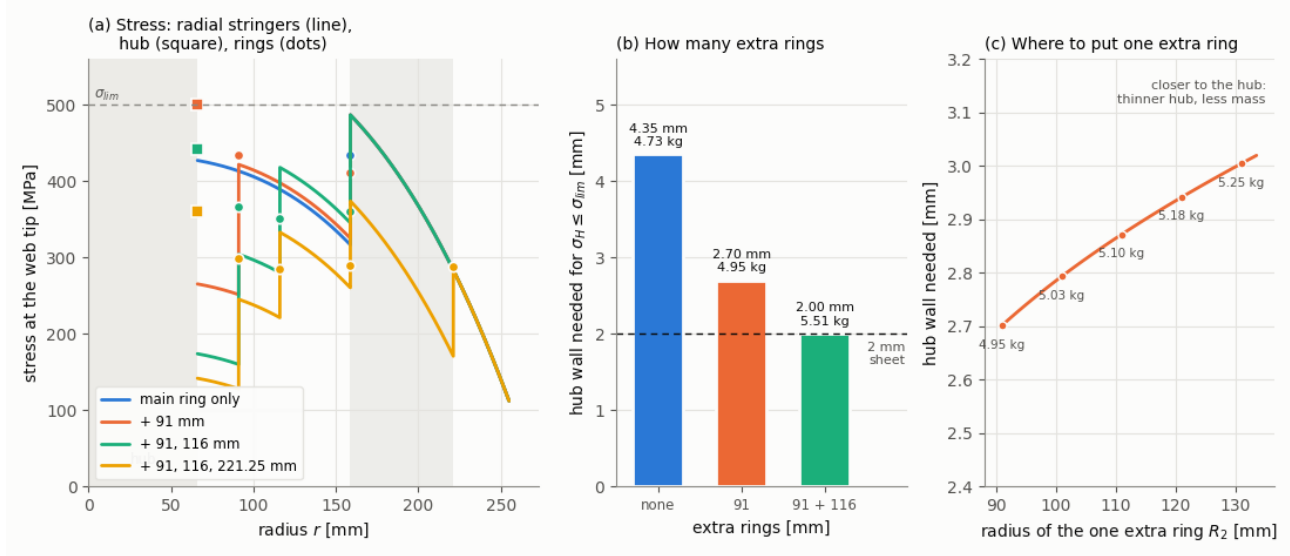


Figure 13: (a) Stress at the web tip along the radial stringers for the variants of Table 14; squares: hub, dots: rings. (b) Hub wall needed for $\sigma_H \leq \sigma_{lim}$ with no, one and two extra rings. (c) Hub wall needed with one extra ring over its radius R_2 (labels: mass of the layout).

Table 14: Centre variants ($n_R = 16$, $h = 30$ mm, hub at 66 mm, main ring at 158.75 mm; plate neglected).

Extra rings [mm]	t_h [mm]	σ_H [MPa]	σ_{end} [MPa]	max. σ_C [MPa]	w_{max} [mm]	m_s [kg]
–	4.35	500.0	427.0	433.5	2.46	4.73
91	2.70	500.0	265.2	433.8	2.36	4.95
91, 116	2.00	442.5	173.8	367.0	2.11	5.51
91, 116, 221.25	2.00	360.7	141.6	298.8	1.70	6.84

Results.

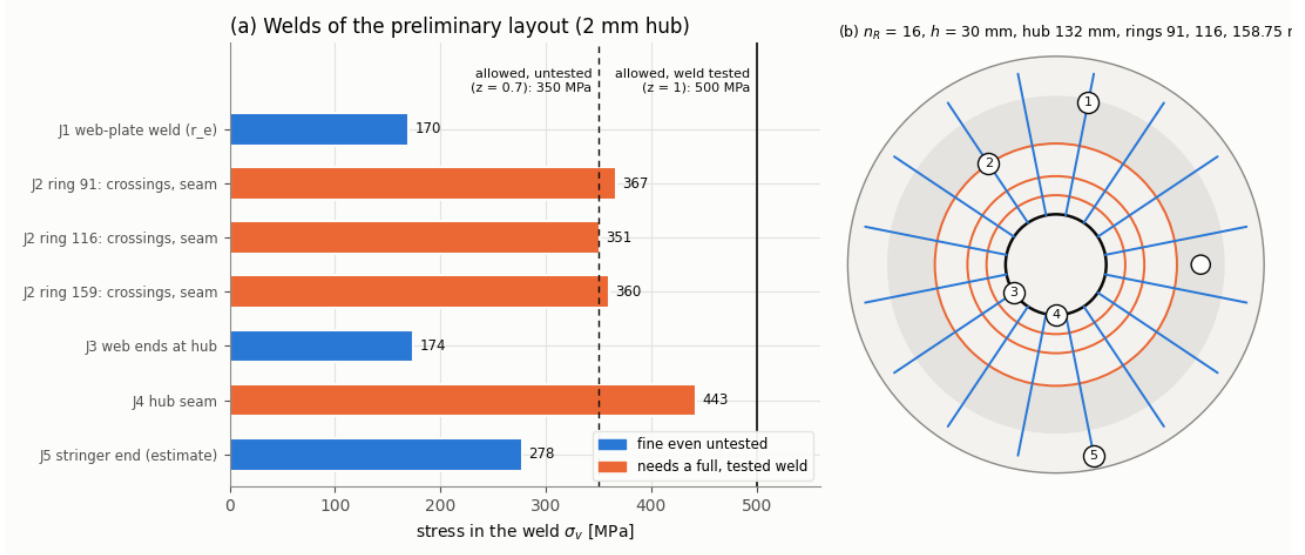


Figure 14: (a) Stress in the welds of the preliminary layout (2 mm hub), compared with the allowed stress for a tested full-penetration weld ($z = 1$, 500 MPa) and for an untested weld ($z = 0.7$, 350 MPa); (b) preliminary layout with the positions of the joints.

Table 15: Joints of the preliminary layout with 2 mm hub ($\sigma_{lim} = 500$ MPa, $a_w = 2$ mm).

Joint	Load	σ_v [MPa]	$z = 1$	$z = 0.7$
J1 web-plate weld (r_e)	shear flow, load introduction	169.6	0.34	0.48
J2 ring 91 mm: crossings, seam	ring moment	367.0	0.73	1.05
J2 ring 116 mm: crossings, seam	ring moment	351.0	0.70	1.00
J2 ring 158.75 mm: crossings, seam	ring moment	360.2	0.72	1.03
J3 web ends at the hub	stringer moment, shear	174.0	0.35	0.50
J4 hub seam	hub moment	442.5	0.89	1.26
J5 stringer end (estimate)	shear flow, end force	277.7	0.56	0.79

- Extra rings relieve the centre: with one ring at 91 mm the hub wall drops from 4.35 to 2.70 mm and the stress at the web ends from 427 to 265 MPa; with two rings (91 and 116 mm) a normal 2 mm ring is enough as hub and the web ends see 174 MPa. The rings sit as close to the hub as the spacing allows, because their effect hardly depends on the radius while their mass grows with it (Fig. 13c).
- The largest stress in the radial stringers (487 MPa, just outside the main ring) is not changed by the inner rings; only a ring further out (221.25 mm) lowers it.
- Chosen layout: hub 132 mm diameter, rings at 91, 116 and 158.75 mm, 16 radial stringers from 66 to 255 mm, all from 2 mm sheet with $h = 30$ mm; $w_{max} = 2.11$ mm, $m_s = 5.51$ kg. The plate inside the hub carries about 59 MPa.
- Web-plate weld (J1) and web ends at the hub (J3) are uncritical, also with $z = 0.7$. The rings and the hub carry their full bending stress through crossings and seams (J2, J4): utilisation 0.70–0.89 with $z = 1$ and 1.00–1.26 with $z = 0.7$. The laser welds cannot be inspected, so $z = 0.7$ applies in the calculation. The layout is kept; the pressure tests (water at low pressure, LN2 at operating pressure) serve as validation of these welds (decision of the team).
- Stringer ends (J5): the estimate gives 278 MPa in the weld; the plate next to the end carries at least the smeared 161.5 MPa (Eq. (56)), the local peak follows from the FE models (Secs. 2.5.5, 2.5.7).
- Local bending of the ring segments between the radial webs and the stresses at the joints follow from the FE models (Secs. 2.5.5, 2.5.7). The Ansys model led to a 5 mm hub wall (Sec. 2.4.5).

2.5.5 FE Check with Beam Elements

Goal. Check the hand models for the layout with 2 mm hub (Sec. 2.5.4) with a discrete model, in which the stringers sit at their real positions and the opening is included, and find what the hand models cannot cover.

Model.

- Plate: DKT elements of Sec. 1.5, simply supported at a as in Sec. 1, $p = 0.7$ MPa.
- Stringers: Euler–Bernoulli beams on the plate mid-plane along the element edges, stiffness EI_w ($h = 30$ mm, $t_s = t_h = 2$ mm), Eq. (41). Rings and hub are polygons of straight beams; torsional stiffness is neglected (open webs).
- Mesh: polar, all stringer lines are element edges, m_f elements per field in circumferential direction; O-grid around the opening; unstructured inside the hub (Fig. 15).
- Output: web tip stress at both ends of each beam element; plate surface stress as largest principal stress from nodal averages; the value 15 mm from a junction is the largest one on a circle of this radius around the beam end.
- Linear elastic and symmetric to the mid-plane: the reversed pressure gives the same values with opposite sign.
- Not contained: the webs as plates (local bending, buckling), the weld geometry and the real load introduction at the joints. The beams are joined to the plate only at nodes, so a beam end passes its moment as a point load into the plate.

Formulation. Beam element of length L and direction $\mathbf{e}$, nodal DOFs as for the plate, $\mathbf{u}_i = [w_i, (\partial w / \partial x)_i, (\partial w / \partial y)_i]^T$:

$$\mathbf{k}_b = \mathbf{T}^T \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \mathbf{T}, \quad \mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & e_x & e_y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & e_x & e_y \end{bmatrix}, \quad (75)$$

$$\sigma_t = \frac{EI}{W_t} |\kappa_{1,2}|, \quad \kappa_1 = \frac{6(w_2 - w_1) - 2L(2\theta_1 + \theta_2)}{L^2}, \quad \kappa_2 = \frac{6(w_1 - w_2) + 2L(\theta_1 + 2\theta_2)}{L^2}, \quad (76)$$

with the slope along the beam $\theta = \mathbf{e} \cdot \nabla w$. The beam matrices are added to the plate matrices of Sec. 1.5.

Verification.

- Plate alone, without opening: $w_{max} = 26.410$ mm (Eq. (14): 26.413 mm).
- Plate with hub and rings only (rotationally symmetric): $w_{max} = 6.725$ mm, ODE (Eq. (55), $S_R = 0$) 6.727 mm; ring stresses 46.5, 108.2, 278.5 and 779.6 MPa in both.
- Mesh study $m_f = 4 \dots 14$ (1643 to 16410 nodes): the global values change by less than 1.1 % (Fig. 17b).

Hand values for the comparison. Opening with Eq. (57) and the force method with hub at r_0 :

$$s_1 = \sigma_\varphi + \sigma_{f,m} = 64.0 + 14.4 = 78.4 \text{ MPa}, \quad s_2 = \sigma_r + \nu \sigma_{f,m} = 66.4 + 4.3 = 70.8 \text{ MPa}$$

$$\sigma_L = |s_1 + s_2| + \frac{2(1 + \nu)}{3 + \nu} |s_1 - s_2| = 149.2 + 0.788 \cdot 7.6 = 155.2 \text{ MPa}$$

The hub wall is estimated in Sec. 2.4.4.

Script. `S2f_FEM_Stiffened.py` builds the mesh, runs the two checks and the mesh study, and compares the result mesh ($m_f = 8$, 5579 nodes) with the ODE (Eq. (55) with hub) and the force method with hub (Table 16, Figs. 15–17).

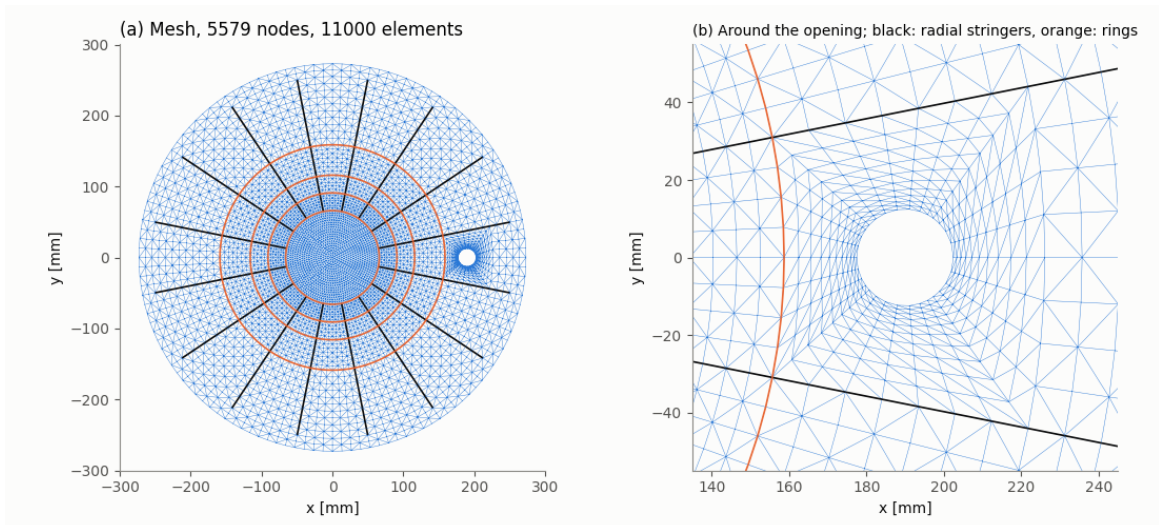


Figure 15: FE mesh of the preliminary layout with 2mm hub ($m_f = 8$). Black: radial stringers, orange: hub and rings. (a) Whole plate, (b) around the opening.

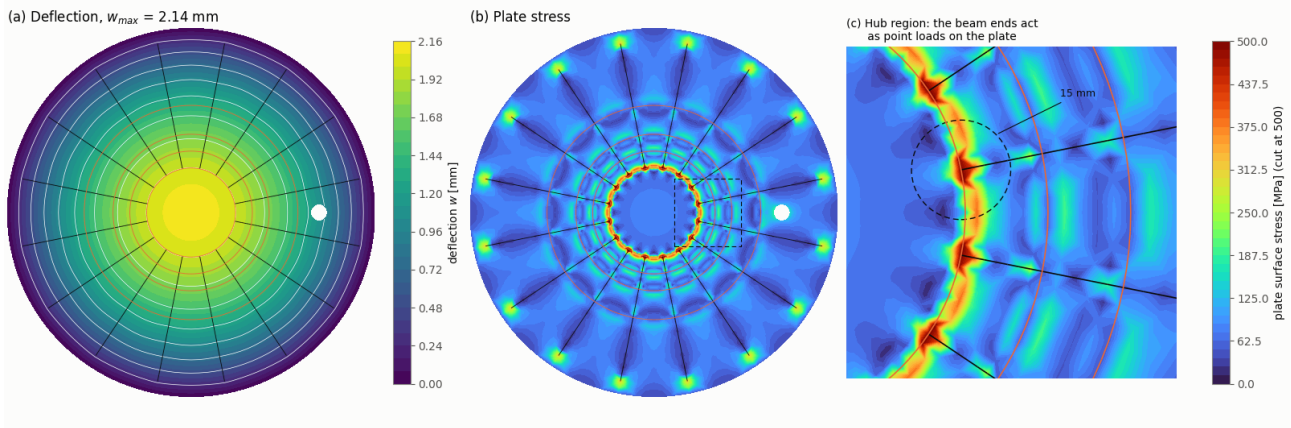


Figure 16: (a) Deflection; (b) plate surface stress, cut at 500 MPa; (c) hub region (dashed box in b), circle: distance of 15 mm from a web end.

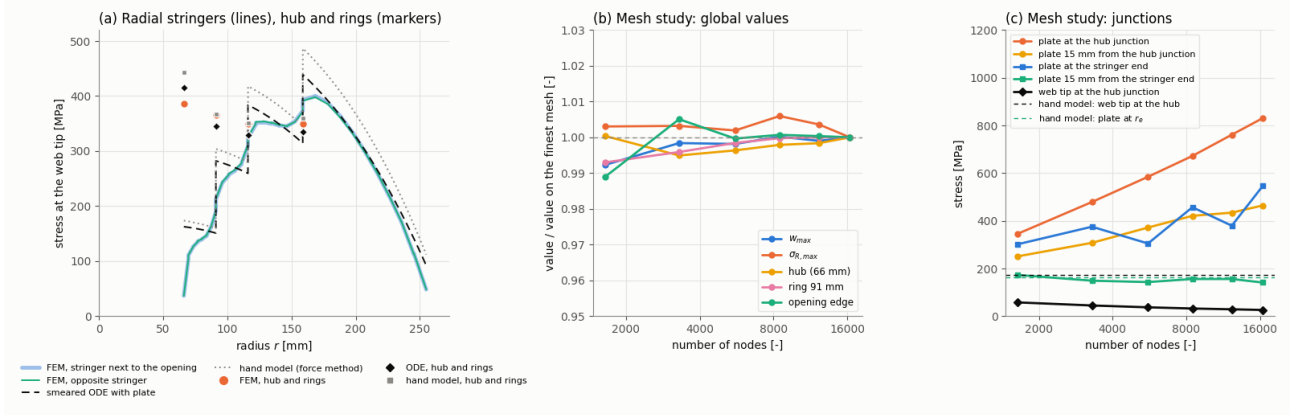


Figure 17: (a) Stress at the web tip along the radial stringers and at hub and rings: FE, ODE with plate and hand model (force method). (b) Mesh study of the global values, (c) mesh study at the junctions; dashed: hand values.

Table 16: FE ($m_f = 8$, maximum along stringer or ring) compared with the ODE with plate and the hand model (force method with hub, plate neglected). Stresses in MPa; *: not converged with the mesh.

Quantity	FE	ODE	hand
w_{max} [mm]	2.14	1.90	2.11
$\sigma_{R,max}$, radial stringers	400.4	439.0	486.5
hub (66 mm)	385.5	415.7	442.5
ring 91 mm	366.0	344.8	367.0
ring 116 mm	349.4	329.1	351.0
ring 158.75 mm	349.9	335.4	360.2
web tip at the hub	37.4*	162.6	173.8
plate at r_0 , middle of the field	95.3	—	78.4
opening edge	177.5	—	155.2
plate 15 mm from the stringer end	143.2	154.9	161.5
plate 15 mm from the hub junction	371.7*	59.9	—

Results.

- Stringers: radial stringers (400 MPa) and hub (386 MPa) stay below the hand values (487 and 443 MPa); the rings agree with the hand model within 3 % and lie 4 % to 6 % above the ODE. The hand model stays the basis for the stringer stresses.
- Deflection 2.14 mm (hand 2.11, ODE 1.90 mm): the radial webs meet the hub only in single nodes, and without torsional stiffness the hub cannot take a point moment, so the web ends act almost as hinges (web tip at the hub 37 MPa instead of 174 MPa). $w_{max} \leq 4$ mm holds with margin.
- Opening: 177.5 MPa at the edge, converged, 14 % above the hand value (155.2 MPa): between the discrete stringers the plate field carries more than the smeared plate plus the strip of Eq. (57) (95.3 against 78.4 MPa at r_0). The opening does not govern.
- Stringer ends: 15 mm from the end the plate carries 141 MPa to 156 MPa ($m_f \geq 6$), as the hand value 161.5 MPa (Eq. (56)). At the end node itself the value grows with the mesh (point load), and the web tip stress at r_e goes to zero, as at a free cut end.
- Hub junction: the FE model is the limit in which the end moment of the radial webs goes into the plate; there the plate stress 15 mm from the junction rises from 251 to 465 MPa with the mesh and at the node from 346 to 831 MPa. The hand model is the other limit, in which it goes into the hub wall; then the 16 web forces bend the 2 mm wall in the plate plane, 631 MPa at the web plus 244 MPa hoop stress (Eq. (72)). The real structure lies between the two. In both limits the local stress at the hub junction reaches or exceeds σ_{lim} ; it is the most critical local point of the layout and is checked in the Ansys model (Sec. 2.5.7).
- The beam model cannot resolve the joints. Its values at the junctions (Fig. 17c) are not used for the design.

2.5.6 Final Layout

S2g_FinalDesign.py evaluates the final layout: the force method with hub for $t_h = 2$ and 5 mm with the joints and the hub wall (Sec. 2.4.5), the ODE with the chamfered ends (Eq. (73)) for the simple support and for the edge restraint by the tank wall (Eq. (58)), and the pressure at room temperature (Sec. 2.4.6); see Fig. 18 and Table 6.

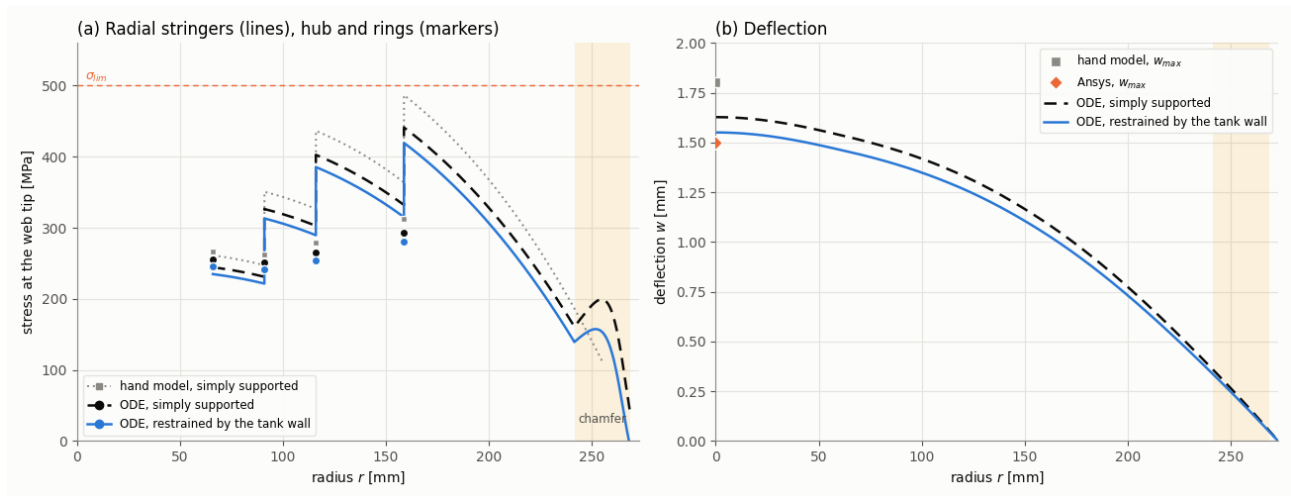


Figure 18: Final layout. (a) Stress at the web tip along the radial stringers and at hub and rings, (b) deflection; shaded: chamfer at the stringer ends.

Results.

- The edge restraint by the tank wall lowers the deflection from 1.63 to 1.55 mm and the stresses in stringers, hub and rings by 4 to 5%; Ansys gives 1.5 mm. For the layout with 2 mm hub the same comparison gives 1.81 mm against 1.92 mm in Ansys.
- Along the chamfer the plate takes over the moment gradually. The stress at the web tip rises to at most 199 MPa (simply supported) and 157 MPa (tank wall); the global plate stress at the stringer ends drops from 157 MPa (straight end) to below 70 MPa.

2.5.7 Ansys Model of the Complete Tank

Model.

- CAD model of the complete tank: bulkhead with stringers, tank wall and domes. The bulkhead is joined to the tank wall by a bonded contact; the fixed support is at the lower end of the LCH4 tank.
- Load: 0.7 MPa on the LOX side, i.e. on the upper face of the plate including the cross-sections of the stringers, on the tank wall and on the dome, as in a closed tank. The edge of the plate is thereby restrained by the compliance of the wall, as in Eq. (58).
- Linear elastic. Small fillets at the joints, coarsely meshed.

Layout with 2 mm hub and straight ends. $w_{max} = 1.92$ mm (ODE with the edge restraint by the tank wall: 1.81 mm). The largest stress, 826 MPa, lies at the upper edge of the hub wall at the radial webs, with a band of about 520 MPa to 650 MPa along the edge between the webs (Fig. 19). The deformed shape shows the mechanism of Eq. (72): the web forces bend the hub wall in the plane of the plate between the webs. The estimate of Sec. 2.4.4 (875 MPa) lies about 6% higher, as it neglects the support of the wall by the plate.

Changes.

- Hub wall 5 mm instead of 2 mm (Sec. 2.4.5). In the following run the largest stress moved to the corners of the straight stringer ends (575 MPa as a point value at a sharp corner, about 200 MPa to 300 MPa a few millimetres away).
- Stringer ends with the 45° chamfer of Eq. (73); the stringer mass stays practically the same.

Final design. $w_{max} = 1.5$ mm (ODE with the edge restraint by the tank wall: 1.55 mm). The peak at the stringer ends has disappeared. The largest stresses, 480 MPa to 510 MPa, now lie at the fillets of the three rings next to the welds, where the ring webs tend to deflect sideways (Fig. 20). The fillets in the model are small and coarsely meshed; the real value depends on the size of the weld. Away from these points the stresses stay in the range of the hand values (Table 6).

2.6 Derivations and Sources

The location in a source is given as printed page, with the PDF page of the library file in parentheses; a key without location refers back to the passage cited just before. Steps that are not taken from the literature but derived here are marked as such.

2.6.1 Mechanical Description

Constraints from the opening. Eqs. (24) and (25) keep every stringer weld outside the zone in which the disturbance of the opening has not yet decayed. In bending, the stress around the hole has practically returned to the value of the undisturbed plate at about $2.5 r_L$ from the hole centre [NACA740], with the hole radius $r_L = d/2$. The distance is measured from the hole centre to the centre line of the stringer; the welds lie within the 2 mm web. For the rings this gives $|R_j - r_0| \geq 2.5 r_L$.

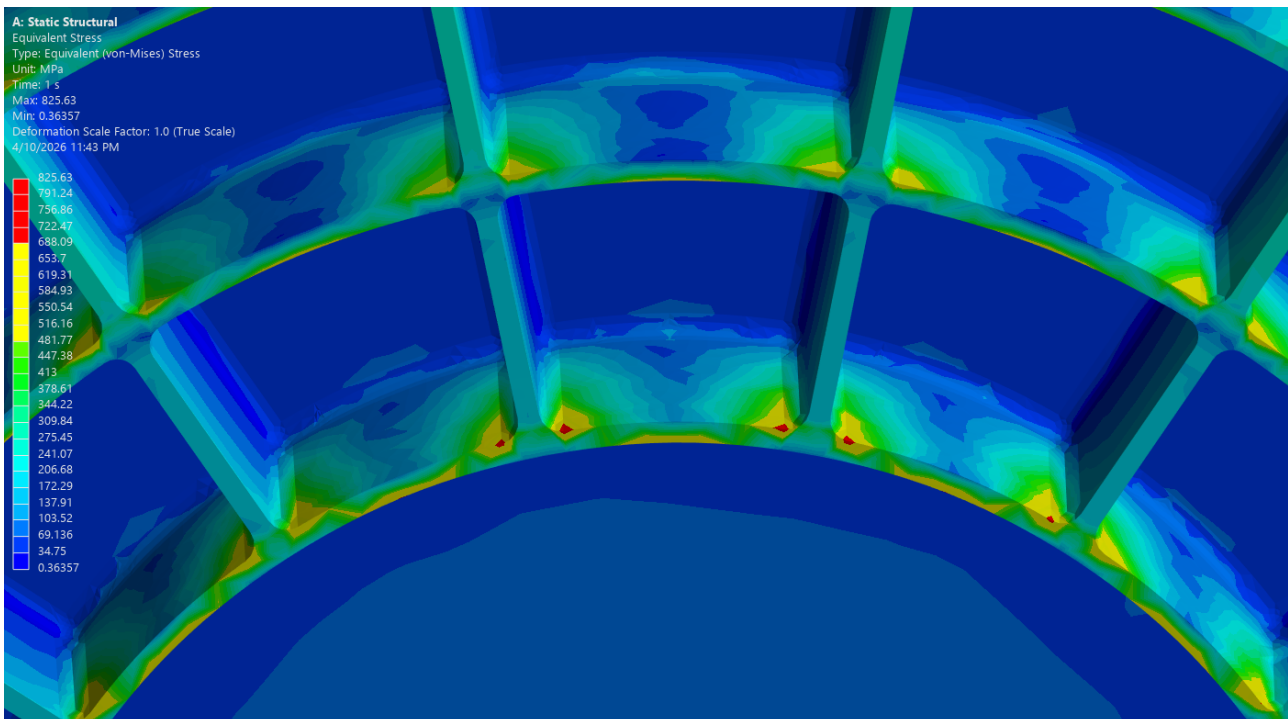


Figure 19: Ansys, layout with 2 mm hub: von Mises stress at the hub and the inner rings (true scale).

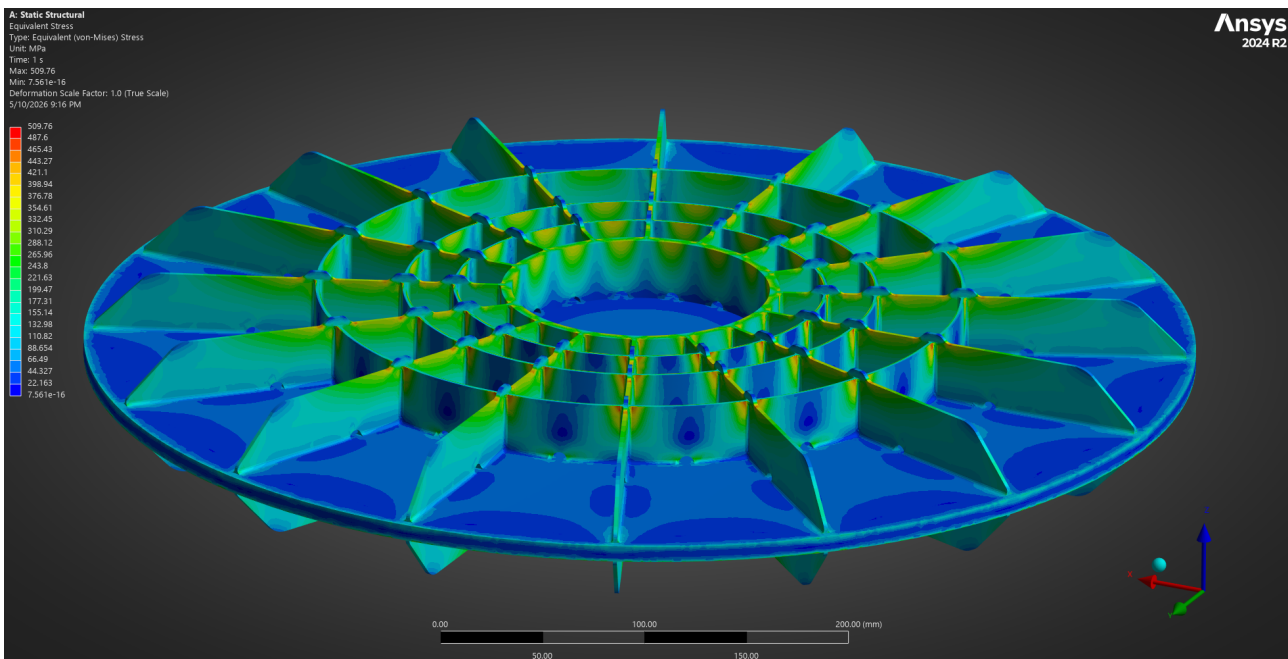


Figure 20: Ansys, final design (5 mm hub, chamfered stringer ends): von Mises stress for 0.7 MPa on the LOX side.

For the radial stringers the field width at r_0 must hold this distance on both sides: $2\pi r_0/n_R \geq 2 \cdot 2.5 r_L$, $n_R \leq 2\pi \cdot 190/62.5 = 19.1$ (derived here).

This clearance is a different rule from the distance $2.5d = 5r_L$ between hole centre and plate edge in Eq. (15). That one is the validity limit of Goodier's solution for the infinite plate; this one only keeps the welds out of the region of raised stress around the hole. Both values are compared in Sec. 1.6.5.

Symmetric cross-section. A plate that is symmetric about its mid-plane shows no coupling between bending and in-plane deformation [W]. For a stiffener that is symmetric to the plate mid-plane, the centroidal axis and the principal axis of the stiffener lie in this plane; the effective width then loses its influence, and the stiffener stiffness is computed from the stiffener profile with respect to the plate plane alone [W]. The stiffnesses of plate and stringers therefore add. The torsional stiffness of a stiffener is only significant for closed profiles [W]. The full stiffness of the stiffener is only effective with a rigid connection to the plate [W], which the continuous welds provide.

Cross-section values. Eq. (26) follows from integrating over both webs, which extend from $|z| = s_P/2$ to $|z| = s_P/2 + h$ [TM2]:

$$I_w = 2t_s \int_{s_P/2}^{s_P/2+h} z^2 dz = \frac{2t_s}{3} \left[\left(\frac{s_P}{2} + h \right)^3 - \left(\frac{s_P}{2} \right)^3 \right] = \frac{t_s}{12} \left[(2h + s_P)^3 - s_P^3 \right].$$

The first moment of one web is its area $t_s h$ times the distance of its centroid $(s_P + h)/2$, Eq. (28). The stiffener ratio Eq. (29) corresponds to Wiedemann's stiffener value $\gamma_s = EI/(Bb)$ [W] with $\gamma_s = 1$, solved for b .

Beam relations. The elasticity law of the beam $M = EI\kappa$ with $\kappa = -w''$, the stress $\sigma = Mz/I$ and the equilibrium $Q' = -q$, $M' = Q$ are taken from [TM2]. The shear flow in a thin-walled section is $\tau t = Q S/I$ [TM2]; at the junction of web and plate this gives Eq. (33). A stiffener has the same deflection and slope as the plate along its line, and it is loaded by the jump of the Kirchhoff equivalent shear force of the adjacent plate fields [W]; hence the curvature of the radial stringer is the radial curvature of the plate, Eq. (30). Eq. (34) assigns the whole pressure of the sector to the stringer (derived here).

Ring kinematics (derived here). The ring web is rigidly attached to the plate. A point of the web at height z moves with the rotation of the plate normal: radially by $u_r = -z \partial w / \partial r$ and circumferentially by $u_\theta = -z (1/R) \partial w / \partial \theta$. Its circumferential strain is

$$\varepsilon_\theta = \frac{u_r}{R} + \frac{1}{R} \frac{\partial u_\theta}{\partial \theta} = -z \left(\frac{1}{R} \frac{\partial w}{\partial r} + \frac{1}{R^2} \frac{\partial^2 w}{\partial \theta^2} \right) = z \kappa_C.$$

The web strain is linear in z , like a beam with the curvature κ_C , which is the circumferential curvature of the plate (its rotationally symmetric part $-w'/r$ appears in [TM4]). Integrating $E\varepsilon_\theta z$ over the web gives Eq. (37). For rotationally symmetric load, the strain energy of the ring is $\frac{1}{2} EI_w (\varphi/R)^2 2\pi R$. Written as $\frac{1}{2} k_C \varphi^2$ per unit circumferential length, this gives $k_C = EI_w/R^2$, Eq. (38).

Smeared stiffnesses. For a plate with parallel ribs the rib stiffness in rib direction is distributed over the rib spacing, while the stiffness across the ribs depends on the skin only [W]. For the radial stringers the spacing is $2\pi r/n_R$, which gives Eq. (35); for a ring the tributary width is l_C , which gives Eq. (39).

Plate stresses. Eq. (40) is the stress at the plate surface $z = s_P/2$ from the elasticity law of the isotropic plate element [W].

Equilibrium of a plate element (derived here). A disc of radius r carries the load $p\pi r^2$ by the shear force q_r on its circumference: $2\pi r q_r = -p\pi r^2$. The moment equilibrium of a plate element $dr \times r d\theta$ gives $d(r m_r)/dr - m_\varphi = r q_r$, hence Eq. (43). For the unstiffened plate ($D_R = D_C = 0$) and the simply supported edge, the moments of Eq. (3) satisfy this equation, which serves as a check.

Energy. Eq. (41) adds the bending energy of the stringers, $\frac{1}{2} \int EI \kappa^2 ds$, to the energy of the plate used for the FE model in Sec. 1.5 [L]. Both share the deflection w , as required by [W].

Mass. The density of 1.4404 is 8.0 kg/dm^3 [EN10088]. Each stringer metre has the web area $2t_s h$.

2.6.2 Radial Stringers Alone

Diametral strip as a beam (derived here). A strip of the angle $d\theta$ through the centre is a beam whose width grows with r ; it carries the load $p r d\theta$ per unit length and is supported at $r = \pm a$. Per radian, each support takes the load of one half, $p a^2/2$. Cutting at r and taking moments of the outer part [TM2]:

$$m(r) = \frac{p a^2}{2} (a - r) - \int_r^a p s (s - r) ds = \frac{p}{6} (a^3 - r^3).$$

With n_R stringers the moment of one stringer is $m 2\pi/n_R$, which gives Eq. (45); the shear force per radian is $|dm/dr| = p r^2/2$, Eq. (47).

Deflection (derived here). With $-w'' = m/S_R$ [TM2], $w'(0) = 0$ (symmetry) and $w(a) = 0$:

$$w' = -\frac{p}{6 S_R} \left(a^3 r - \frac{r^4}{4} \right), \quad w = \frac{p}{6 S_R} \left[\frac{a^3 (a^2 - r^2)}{2} - \frac{a^5 - r^5}{20} \right],$$

which is Eq. (46); at $r = 0$, $w_{max} = 9 p a^5 / (120 S_R) = 3 p a^5 / (40 S_R)$.

Stress and deflection (derived here). The largest curvature occurs at the centre. Eliminating S_R with Eq. (46) gives

$$\kappa_{max} = \frac{m(0)}{S_R} = \frac{p a^3}{6 S_R} = \frac{20 w_{max}}{9 a^2},$$

and with $\sigma_t = E \kappa z_t$ (Eq. (31)) follows Eq. (48). Solving $w_{max} \leq w_{lim}$ and $M_{R,max}/W_t \leq \sigma_{lim}$ for n_R gives Eq. (49).

2.6.3 Rings and Plate

Ring as a line moment (derived here). A ring with the bending moment M_C (constant over the circumference) is cut into a sector $d\theta$. The moment vectors on the two cut faces point in the radial directions at θ and $\theta + d\theta$; their sum is a moment $M_C d\theta$ about the tangential axis, i.e. about the same axis as the radial moment $r m_r d\theta$ of the plate. Per radian the ring therefore applies the line moment M_C at R , and the moment per radian jumps by this amount. The same follows from the smeared ring stiffness: integrating $D_C \kappa_\varphi$ (Eq. (39)) over the tributary width l_C in Eq. (43) gives the jump $E I_w \kappa_C = -E I_w \varphi / R$ of Eq. (55).

Force method. For an n -fold statically indeterminate system the redundants follow from $\alpha_{r0} + \sum_i X_i \alpha_{ri} = 0$ with $\alpha_{r0} = \int \bar{M}_r M_0 / (EI) dx$ and $\alpha_{ri} = \int \bar{M}_r \bar{M}_i / (EI) dx$ [TM2]. Here the base system is the beam per radian of Sec. 2.3.9 with $M_0 = p(a^3 - r^3)/6$. A unit moment at R_j gives $\bar{M}_j = -1$ for $r < R_j$ and 0 outside, hence $\alpha_{jk} = \int_0^{\min(R_j, R_k)} dr / S_R = \min(R_j, R_k) / S_R$ and $\alpha_{j0} = -\int_0^{R_j} M_0 dr / S_R$, which is the slope of the radial stringers alone at R_j . The ring is elastic: with $X_j = E I_w \kappa_C$ and $\kappa_C = -\varphi / R_j$ (Eq. (38)) its rotation is $\varphi = -c_j X_j$, $c_j = R_j / (E I_w)$. Instead of 0, the right-hand side of the compatibility condition is therefore $-c_j X_j$, which gives Eq. (50) (spring term derived here).

Single ring (derived here). For one ring, $X = -\alpha_{10}/(\alpha_{11} + c_1)$. With $S_R = n_R EI_w/(2\pi)$, R and EI_w cancel, which gives Eq. (51). In the same way all α and c_j in Eq. (50) are proportional to $1/(EI_w)$, so the X_j do not depend on h .

Moments and deflection (derived here). M_0 decreases outwards and jumps upwards by X_j at every ring, so the maximum is either at $r = 0$ or just outside a ring, Eq. (53). The deflection follows from the reduction theorem: the real moment M of the indeterminate system is combined with the moment $\bar{M} = a - r$ of a unit force at the centre on the determinate base system [TM2]:

$$w_{max} = \int_0^a \frac{M(a-r)}{S_R} dr = \frac{3pa^5}{40S_R} - \sum_j \frac{X_j}{S_R} \int_0^{R_j} (a-r) dr,$$

which is Eq. (54).

Plate model (derived here). Inserting $D_R = S_R/r$ (Eq. (35)), $\kappa_r = -\varphi'$ and $\kappa_\varphi = -\varphi/r$ into Eq. (42) gives $M = r m_r = -(rK + S_R)\varphi' - K\nu\varphi$ and $m_\varphi = -K\nu\varphi' - K\varphi/r$; with Eq. (43), $M' = m_\varphi - pr^2/2$, this is Eq. (55). At the centre the symmetry gives $\varphi(0) = 0$; the stringer moment there is finite because opposite stringers act as one beam, so $M(0)$ is unknown. The system is linear and is solved by integrating from the centre twice (load only, and unit curvature at the centre without load) and superposing both so that $M(a) = 0$. For $S_R = 0$ it reduces to the plate of [TM4], for $K \rightarrow 0$ to the beam of Eq. (46).

Edge restraint by the tank wall (derived here). The tank wall acts on the plate edge as a rotational spring with k_M per unit length (Eq. (9)). Its moment opposes the edge rotation, $m_r(a) = k_M \varphi(a)$; for a negative edge slope this is a hogging moment. With $M = r m_r$ this gives Eq. (58). For $k_M \rightarrow 0$ it is the simple support, for $k_M \rightarrow \infty$ the clamped edge $\varphi(a) = 0$. The two integrations of the ODE are superposed with this condition instead of $M(a) = 0$.

Plate at the stringer ends (derived here). Between r_e and a there are no stringers, so the plate carries the whole moment per radian of the base system, $M = p(a^3 - r^3)/6$. At r_e , $m_r = M/r_e$ and $\sigma = 6m_r/s_P^2$ give Eq. (56). The hoop moment of the plate is neglected, which makes the value slightly conservative.

Chamfered stringer ends (derived here). A 45° chamfer lowers the web height by 1 mm per millimetre of length, which gives Eq. (73). Above r_e the chamfer removes a triangle of $\Delta \times \Delta$, below r_e it adds the same triangle; only the strip of height h_{end} over the length Δ is added, $3 \cdot 13.5 = 40.5 \text{ mm}^2$ per web and side or 0.02 kg in total. Along the chamfer the web stiffness decreases gradually, so the plate takes over the moment over the length 2Δ instead of at one point.

Plate fields. Where a skin runs over several equidistant supports, the single fields clamp each other symmetrically [W]. For the long clamped plate strip $w_{max} = pb^4/(384B)$ and $m = pb^2/12$ at the edge [W], with $B = K$; $\sigma = 6m/s_P^2$ gives σ_f in Eq. (57). The moment in the middle of the field is the simply supported value $pb^2/8$ [W] minus the edge moment, $pb^2/24$ (derived here), which gives $\sigma_{f,m}$. In cylindrical bending the moment along the strip is ν times the moment across it [W]. The strip is the limit for long fields; with rings the fields are shorter and the values smaller.

Layout search (derived here). Because the X_j do not depend on h , each layout (n_R and the set of ring radii) gives one required W_t and I_w and therefore one h and one mass; h is a result, not a search variable. All layouts on the grid are evaluated, so the lightest layout for a given h_{max} is the optimum of the grid and not a local one. The lower limit of h follows from Eq. (52): the moment just outside the inner band edge, $M(R_{in}^+) = p(a^3 - R_{in}^3)/6 - \sum_{R_j > R_{in}} X_j$, is only reduced by the rings in the outer band, of which at most two fit with the spacing of 25 mm. Without rings Eq. (48) with $\sigma_t = 500 \text{ MPa}$ and $w_{max} = 2 \text{ mm}$ gives $z_t = 41.9 \text{ mm}$, i.e. $h = 37.9 \text{ mm}$, below which the deflection governs.

2.6.4 Web Buckling

Buckling stress. The buckling stress of a plate strip is $\sigma_{kr} = k_\sigma E(t/b)^2$ [W]. For a strip with one free edge Wiedemann gives $k_\sigma = 0.38$ for a simply supported and $k_\sigma = 1.16$ for a clamped long edge [W]. In the plastic range E is replaced by the effective modulus E_w , which for the clamped strip with a free edge is Eq. (62); the upper limit of the buckling stress is practically $\sigma_{0.2}$ [W]. Flanges that are thicker than the buckling web raise its restraint up to the clamped limit [W].

Rotational spring of the plate (derived here). For a rotation $\theta_0 \sin(\pi x/L)$ of the root line, the deflection $w = \theta_0 y e^{-\alpha y} \sin(\alpha x)$, $\alpha = \pi/L$, of a semi-infinite plate $y \geq 0$ satisfies the plate equation without load [TM4], $w = 0$ and $w_{,y} = \theta_0 \sin(\alpha x)$ at $y = 0$. With $m_y = -K(w_{,yy} + \nu w_{,xx})$ [TM4] the edge moment is $m_y(0) = 2\alpha K \theta_0 \sin(\alpha x)$, i.e. $2\pi K/L$ per plate half and $4\pi K/L$ for both halves, Eq. (59). The decay length is $L/\pi \approx 22$ mm ($h = 40$ mm). Where the field is wider than this, the semi-infinite plate is a fair approximation. Near the centre the fields are narrower; a narrow strip between two webs that rotate together resists with $2K/b$ to $6K/b$ per edge (beam analogy), which is stiffer than the semi-infinite plate. The factor $f = 0.1$ covers the remaining uncertainty.

Ritz solution (derived here). The strain energy is that of the Kirchhoff plate [L] applied to the web, and the work term W is the energy form of the buckling equation with in-plane force $N = \sigma t$ [TM4]. The polynomials $(z/h)^i$, $i = 1 \dots 5$, satisfy $v = 0$ at the root; the free edge needs no condition. For a clamped root the series starts at $i = 2$. Inserting the ansatz and integrating over one half-wave gives the matrices **K** (from $U_w + U_\theta$) and **G** (from W); the smallest eigenvalue over L is the buckling stress. Check: for uniform compression the model reproduces TM4 Tab. 5.1 [TM4] for both edge conditions within 0.6 %; the table values correspond to $\nu = 0.25$, while the design uses $\nu = 0.3$. For long strips and $\nu = 0.3$ it gives the values 0.38 and 1.16 of Wiedemann.

Material curve. The Ramberg–Osgood law $\varepsilon = \sigma/E + \alpha(\sigma_0/E)(\sigma/\sigma_0)^m$ [TM4] with $\sigma_0 = R_{p0.2}$ and $\alpha\sigma_0/E = 0.002$ gives Eq. (61) (derived here). Tests on 316L at 173 K show a yield strength of 860 MPa (hot extruded and annealed material) and a sharp transition from the elastic to the plastic range [Li2023]; the design value 500 MPa is therefore conservative, but the shape of the transition cannot be read from the published curve. Hence the lowest buckling stress for $m = 4 \dots 20$ is used.

2.6.5 Centre Hub and Joints

Hub in the force method (derived here). For $r \geq R_h$ the moment per radian follows from the equilibrium of the outer part as in Eq. (52), independent of the inside. With the plate inside neglected, nothing inside the hub carries moment, so the hub takes the whole $M(R_h)$, Eq. (65). The symmetry condition $\varphi(0) = 0$ of Eq. (50) is replaced by the hub rotation $\varphi_h = -c_h M_h$ (Eq. (38) with the hub wall). Integrating $w'' = -M/S_R$ from R_h gives $\varphi(R_k)$, and with $\varphi(R_k) = -c_k X_k$ Eq. (64); for $R_h \rightarrow 0$ it reduces to Eq. (50). The plate disk inside the hub follows the edge rotation with $w = \varphi_h(r^2 - R_h^2)/(2R_h)$, i.e. $|\varphi_h|R_h/2$ at its centre, plus its own deflection as a simply supported disk (Eq. (5) with $a \rightarrow R_h$); its stress is Eq. (14) with $a \rightarrow R_h$: $6 \cdot 3.3 \cdot 0.7 \cdot 66^2/(16 \cdot 8^2) = 59$ MPa.

Why a central tube needs $n_R/(2\pi)$ times the stringer moment (derived here). At the hub the moment per radian jumps from about zero to M_h ; this jump is the hub's own bending moment (ring as a line moment, see above). One stringer carries $(2\pi/n_R)M_h$, so the hub carries $n_R/(2\pi)$ times the moment of one stringer, independent of its radius; with the same cross-section its stress is $n_R/(2\pi) = 2.55$ times higher.

Ring crossings (derived here). Between two radial webs a ring segment carries the ring moment X at both ends; its moment vector points along the ring cross-section, i.e. radially. At a radial web the end moments of the two adjacent segments include the angle $2\pi/n_R$. Their difference passes through the web and the two welds as $\sigma_\perp = X/W_t$; their resultant $2X \sin(\pi/n_R)$ points in circumferential

direction and is a bending moment of the radial stringer, Eq. (69). For large n_R it equals the smeared jump $X 2\pi/n_R$ of Eq. (52).

Slot at a ring crossing (derived here). In the final construction the ring is continuous, and the radial web has a slot of the web thickness over the height of the ring, open towards the plate, and runs on above the ring as a small raised bridge. Over the height of the slot the web of the radial stringer is interrupted; its normal stress passes through the welds and across the ring web, in compression also by contact. The moment of the radial stringer is larger on the outer side of the ring (by $X_j 2\pi/n_R$), so the welds carry at most the stress at the web tip just outside the ring, Eq. (71). The ring moment needs no weld at the crossing.

Weld stresses. The shear flow is Eq. (33); the transverse load of one web root is half the line load dQ_R/dr of the stringer (derived here). Normal and shear stress in the weld are combined with the distortion energy hypothesis [TM2]. For fillet or partially penetrated welds without non-destructive testing, a weld factor of at most 0.7 is used in [EN13445-3].

Stringer end (derived here). At a straight end the web force must pass into the plate. Its resultant is $F_w = t_s \int \sigma_e(z/z_t) dz$ over the web, Eq. (70). Spreading it over the length h is an assumption that the FE model has to confirm.

2.6.6 Beam-Plate FE Model and Hub Wall

Beam element (derived here). Along the beam the deflection is interpolated cubically with the end values $w_1, \theta_1, w_2, \theta_2$. The bending energy $\frac{1}{2} \int_0^L EI (w'')^2 ds$ of Eq. (41) then gives the 4×4 matrix of Eq. (75), and w'' at $s = 0$ and $s = L$ gives κ_1, κ_2 of Eq. (76). The slope along the beam is $\theta = e_x \partial w / \partial x + e_y \partial w / \partial y$, which is the transformation **T**. The beam lies on the plate mid-plane without offset because the stringers are symmetric to it (Sec. 2.6, symmetric cross-section).

Ring as a polygon (derived here). A chord of a ring of radius R spans the angle $\Delta\theta$, its length is $L = 2R \sin(\Delta\theta/2)$. For a rotationally symmetric slope $\nabla w = \varphi \mathbf{e}_r$ both ends have the same w , and the chord direction includes the angle $90^\circ \mp \Delta\theta/2$ with the radius at its ends, so $\theta_{1,2} = \mp \varphi \sin(\Delta\theta/2)$. Eq. (76) gives $\kappa_1 = \kappa_2 = 2\varphi \sin(\Delta\theta/2)/L = \varphi/R$, i.e. the ring moment $EI_w \varphi/R$ of Eq. (55), independent of the number of chords. The second check confirms this.

Junction values. Where a beam ends, or where its moment changes at a node, the moment passes into the plate at one point. The plate moments of a concentrated load grow without limit towards the load point, so the values at and near these nodes depend on the mesh (Fig. 17c); they are a property of the model, not of the structure.

Hub wall between the webs (derived here). The web force F at the hub follows from σ_{end} as F_w in Eq. (70). For a ring with n_R equal radial loads F the segment between two mid-points ($\pm\alpha$ from the load) carries at its ends, by symmetry, no shear force, the normal force N_0 and the moment M_0 . Radial equilibrium gives $F = 2N_0 \sin \alpha$, and at the angle ψ from the load $M(\psi) = M_0 + N_0 R [1 - \cos(\alpha - \psi)]$. Load and mid-section do not rotate (symmetry), so with the force method [TM2] $\int_0^\alpha M d\psi = 0$, hence $M_0 = -N_0 R (1 - \sin \alpha / \alpha)$ and $M_L = M(0) = N_0 R (\sin \alpha / \alpha - \cos \alpha)$, which is Eq. (72); $N = N_0 \cos \alpha$. As a check, the hoop stress $N/(ht_h) = 244 \text{ MPa}$ equals the mean of the smeared hub stress over the web, $442.5 \cdot (4 + 34)/(2 \cdot 34) = 247 \text{ MPa}$. The lower edge of the wall is welded to the plate, which holds it radially; the estimate neglects this and is therefore on the high side.

2.6.7 Allowed Pressure at Room Temperature

Linearity. In the elastic models all stresses and deflections are proportional to the pressure. The allowed pressure at room temperature keeps the ratio of the largest stress to the yield strength, $\sigma(p_{RT})/R_{p0.2}(\text{RT}) = \sigma(p)/\sigma_{lim}$, which gives Eq. (74) (derived here).

Temperature dependence. [Li2023] tested a commercial 316L, hot extruded and annealed at 1040 °C [Li2023], at 297, 173, 50 and 15 K (two specimens each, with a small difference). The yield strength rises from 660 MPa at 297 K to 860 MPa at 173 K and 990 MPa at 50 K [Li2023]. The operating temperature 103 K lies between two measured values; the linear interpolation between them is an assumption (derived here), and the bounds follow from the two measured values themselves.

3 Manufacturing

The bulkhead consists of the plate and the stringer grid, which is the same on both sides of the plate. Each of the 16 radial stringers of Sec. 2 consists of one radial web on each side, which gives 32 radial webs. All parts are made of 1.4404 (316L) (Table 17, geometry in Table 5). The radial webs are cut from 2 mm sheet by waterjet or plasma cutting. The rings are bent from 2 mm sheet and welded to cylinders, the hub is rolled from 5 mm strip. Fig. 21 shows the final design.

Table 17: Parts of the bulkhead (per bulkhead, both sides; per side 16 radial webs, one hub and one of each ring).

Part	Number	Blank	Note
Plate	1	8 mm sheet, 546 mm diameter	opening at $r_0 = 190$ mm
Radial webs	32	2 mm sheet, 201.5×30 mm	$r = 67 \dots 268.5$ mm, chamfer, 3 slots
Ring 91 mm	2	2 mm strip, 572×30 mm	bent, one seam
Ring 116 mm	2	2 mm strip, 729×30 mm	bent, one seam
Ring 158.75 mm	2	2 mm strip, 998×30 mm	bent, one seam
Hub	2	5 mm strip, 405×30 mm	rolled, one seam

The lengths of the rings are the developed lengths of the centre line, $2\pi R_j$. The hub strip has the length of its mean fibre, $2\pi \cdot 64.5$ mm (mean radius of the wall, $r = 62 \dots 67$ mm), and assumes pre-bent ends; on a roll bending machine without pre-bending, the straight ends need an allowance that depends on the machine.

Details. At each crossing with a ring the radial web has a rectangular slot of 2 mm width over the height of the ring, open towards the plate. Above the ring the web runs on as a small raised bridge, so that each radial web stays one piece; rings and radial webs have the same height. The rings are continuous and stand in the slots. Half holes of 3 mm at the plate in the webs let LOX drain from the fields between the stringers; they lie in regions of low stress and are included in the final Ansys model.

Assembly. The grid of one side is assembled as follows:

1. Stand the rings concentrically.
2. Push the radial webs onto the rings from above, so that the bridges rest on the rings.
3. Align the parts with a 3D-printed fixture (PLA) and place the hub in the centre.
4. Tack weld the grid; the fixture is only used up to this step.
5. Place the grid on the plate, with the open side of the slots on the plate, and tack it to the plate.
6. Weld all joints.

The grid on the other side of the plate is made in the same way.

Joints.

- The welds are made with a laser welding unit by THEO in a dedicated welding cell; the heat input is concentrated. Every edge is welded.
- All stringers are laser welded to the plate, through the web ($a_w = 2$ mm); the hub seam goes through the 5 mm wall.
- The rings are welded to the radial webs in the slots, the radial webs to the hub. At the crossings the welds are built up to a fillet; with a small fillet these points carry the largest stresses of the design (480 MPa to 510 MPa, Sec. 2.5.7).

- The welds in the slots carry the stress of the radial stringers, up to 487 MPa at the main ring (J6 in Table 10).
- The welds cannot be inspected; the calculation uses the weld factor $z = 0.7$ (Table 10).

Tests.

1. Water test at room temperature with 0.47 MPa; the expected deflection at the centre is about 1.0 mm (Sec. 2.4.6).
2. LN2 test with the full pressure difference of 0.7 MPa, with the bulkhead cold.
3. Test with the real propellants.

The pressure difference of 0.7 MPa is only allowed with the bulkhead at operating temperature; at room temperature it must stay at or below 0.47 MPa.

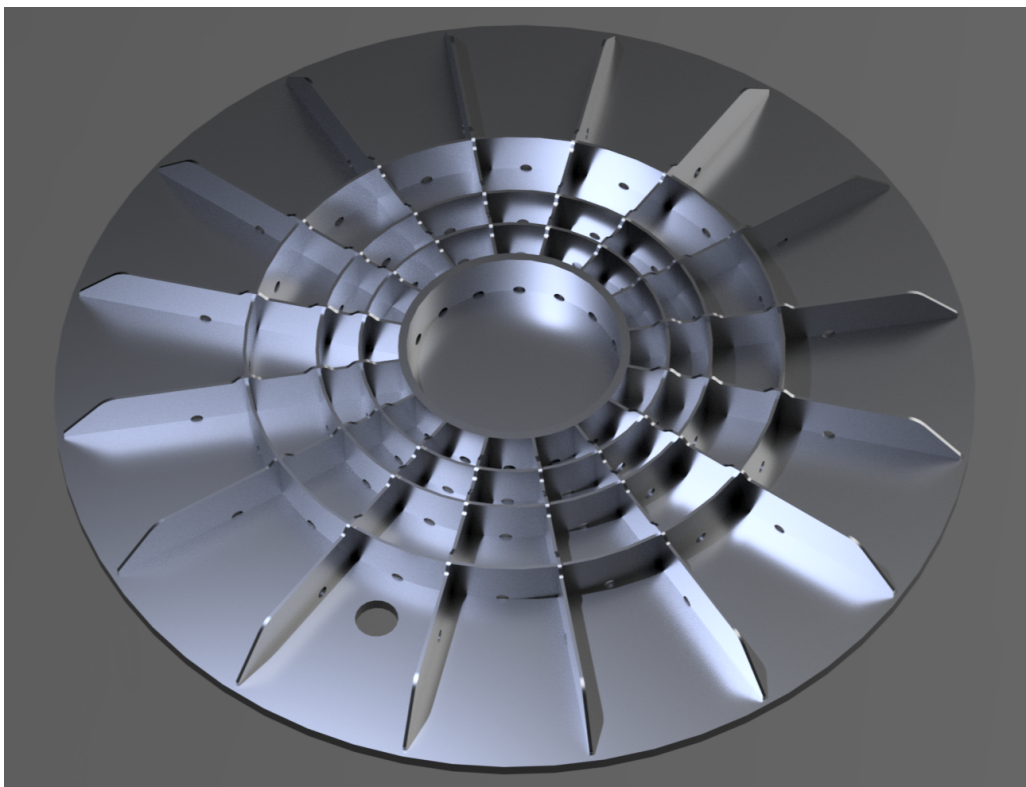


Figure 21: CAD model of the final bulkhead.

References

- [EN10088] DIN EN 10088-1:2024-04, *Nichtrostende Stähle – Teil 1: Verzeichnis der nichtrostenden Stähle*, April 2024. Library file 02_DIN/DIN EN 10088-1.
- [EN13445-3] DIN EN 13445-3:2021-12, *Unbefeuerte Druckbehälter – Teil 3: Konstruktion*; German version EN 13445-3:2021, December 2021. Library file 02_DIN/DIN EN 13445-3.
- [L] M. Link: *Finite Elemente in der Statik und Dynamik*, 4th ed., Springer Vieweg, Wiesbaden, 2014. Library file 01_017.
- [Li2023] S. Li, P. J. Withers, S. Kabra, K. Yan: The behaviour and deformation mechanisms for 316L stainless steel deformed at cryogenic temperatures, *Materials Science & Engineering A* 880 (2023) 145279, doi:10.1016/j.msea.2023.145279.
- [NACA740] C. Dumont: *Stress Concentration around an Open Circular Hole in a Plate Subjected to Bending Normal to the Plane of the Plate*, NACA Technical Note No. 740, Washington, 1939. Library file NASA_19930081505.
- [TM2] D. Gross, W. Hauger, J. Schröder, W. A. Wall: *Technische Mechanik 2: Elastostatik*, 13th ed., Springer Vieweg, Berlin, 2017. Library file 01_010.
- [TM4] D. Gross, W. Hauger, P. Wriggers: *Technische Mechanik 4: Hydromechanik, Elemente der Höheren Mechanik, Numerische Methoden*, 11th ed., Springer Vieweg, Berlin, 2023. Library file 01_046.
- [W] J. Wiedemann: *Leichtbau – Elemente und Konstruktion*, 3rd ed., Springer, Berlin, 2007. Library file 01_028 (printed page = PDF page – 33).